Money-Sharing and Intermediation in Networks

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Abstract

We study the problem of transmission of a divisible resource (such as money) to agents via a network of intermediaries. The planner has preferences over the different allocations of the resource to the agents. Although the planner is not directly linked to the agents, it can connect via a group of intermediaries. Intermediaries may differ in the *types* of agents they can reach, as well as the *quality* in which they can reach agents.

The planner solicits bids from intermediaries to use their links and applies this information to select which intermediaries to contract for the transmission of the resource. The intermediaries choose the fees in order to maximize the amount paid by the planner. The planner picks the allocation that maximizes his utility over the resource allocated to the agents. A game theory model is constructed to analyze the strategic behavior of the planner and the intermediaries. We present the necessary and sufficient conditions for the existence of a Subgame Perfect Nash Equilibrium (SPNE) and Efficient SPNE, where the intermediaries used by the planner charge zero cost. This equilibrium depends on the network configuration, the quality in which the intermediaries reach the agents, and the preferences of the planner.

Multiplicity of SPNEs often occur. We also present a robustness of the SPNE, whereby the intermediaries who are not used by the planner charge zero cost (Robust-SPNE). The necessary and sufficient conditions for the uniqueness of an Efficient Robust-SPNE are provided. Comparative statics, with respect to the addition of intermediaries, are given. Finally, when the planner has the ability to change the quality in which the intermediaries connect to agents, we characterize the large class of networks that induce an efficient SPNE.

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1 Introduction

There are many markets for which intermediaries play an essential role. The most common markets for which intermediaries are critical include the transmission of goods and resources to agents. For instance, the allocation of government resources (such as money) to agents often requires the use of private for-profit companies, intermediaries, that are more closely connected to the agents than the government agency. In this way, the government agency can more effectively target their agents than without intermediaries. Additionally, this top-down structure of the network provides opportunity for competition between intermediaries with the potential for added benefits. Such benefits, however, are largely dependent on the way in which the intermediaries are connected to the agents (including the individual connection, as well as the quality of the connections) and the type of resource (i.e., divisible vs indivisible) that is distributed to the agents.

Although much attention has been paid to the case of intermediation for indivisible goods, few studies focus on intermediation for divisible goods and resources. Herein, we describe a model where a planner is interested in transmitting a divisible resource to agents (such as money). The planner has preferences over the different allocations of the resource to the agents. Although the planner is not directly linked to the agents, it can do so via a group of intermediaries. Intermediaries may differ in the *types* of agents they can reach, as well as the *quality* in which they can reach the agents. We focus in the case of exogenous quality of intermediaries. The quality, can be interpreted as the effective transmission of the resource from the intermediaries to the agents. This is represented by the total amount of the resource that an intermediary sends to the agents per unit of resource received, as well as by the proportions in which every agent receives a resource relative to another from a given intermediary. Thus, for instance, two intermediaries connected to the same group of agents may be very different from the planner's perspective, since the may transmit different amounts to the agents, and even make different distributions between them.

We study the case of perfect information, where the planner and intermediaries are aware of the preferences of the planner and the connections and quality of the intermediaries. The planner solicits bids from intermediaries to use their links and applies this information to select which intermediaries to contract for the transmission of the resource. We used a game theoretical approach to model the behavior of the planner and intermediaries.

Specifically, in the first stage of our game, intermediaries independently and simultaneously report their fees for providing the planner with access to the agents. In the second stage, the planner then selects the appropriate intermediaries and amounts of the resource allocated to each of them for transmission to agents. The intermediaries who are not selected do not get paid. The ultimate goal of the intermediary is to be contracted and maximize the price paid by the planner. The goal of the planner is to distribute as much resource to the agents in a way that maximizes the planner's preferences. We use a Subgame Perfect Nash Equilibrium (SPNE) to describe the result of strategic behaviors between the planner and intermediaries. The biding equilibrium price of intermediaries depends on the utility function of the planner, the structure of the network, as well as the quality of the connection from the intermediaries to the agents.

An application of our game theoretical model is the transmission of advertising money in companies. A company looking to promote their product can use different media (the intermediaries) to reach the advertising target of their product; such intermediaries include TV channels, radio stations, Internet websites and newspapers. The quality of the connections is relevant because, within the media, there are different channels that target to specific demographics of agents and may influence the planner's objective differently. For instance, two local TV stations based in the same city may be connected to all agents in the city, but the audience may be more biased based on demographics or political preferences —e.g. Fox News and CNN reach the same audience, but they target their programming to attract more conservative or liberal viewers, respectively. Nowadays, the printed version of newspapers are read heavily by older people instead of younger people, and the proportions of older to younger readers are typically available to potential purchasers of advertisements. Therefore, it is in the interest of the planner to choose the media channel that best align to his preferences.

Alternatively, consider the case of government contracting. For instance, the allocation of government's money to people in need via charities. The government may decide to send the money via charities that will charge an indirect cost for the use of their services. The connections of the charities as well as their quality are exogenous information that the planner cannot control, and they are typically taken into account when making a decision on how to allocate the resources. For instance, charities heavily funded by the government include UNICEF or the Red Cross. While both charities overlap in some of the agents that they serve (e.g. children in need), they also have large difference in their recipients.[1] The quality of the connections of the charities is also important when picking a charity. For instance, inefficiencies happen often in charities and universities, where every dollar spent is often decreased due to indirect cost, which serves to pay for administration.[2] Thus, the planner should care about how their money is distributed to the agents and aligned with its preferences. Our model looks at the case of perfect information, which is also the case in this example, as the priorities and activities of the charities are typically reported by them in advance.[3] As such, the planner can make an informed decision on how its money will transmit by the charities chosen.

Finally, the problem has applications to network flow problems. For instance, when there is ground water that must be distributed to agents via private canals (intermediaries). The planner can decide how to route the water to the canals, but once the water reaches the canal it is distributed to the agents connected to these canals in some fixed proportions that may vary between canals. Conveyance losses are typical in models, and may depend on how far the agents are from the source (Jandoc, Juarez and Roumasset[22] study the optimal allocation of water networks in the presence of these losses). The owners of the canals may charge the planner for the use of their canals, and therefore the planner should consider the trade-offs between allocating goods to cheap canals as opposed to more efficient but expensive canals.

While most of the analysis of the paper studies the case of exogenous quality of the intermediaries. The end of the paper will study the case where the planner can choose the quality of the intermediaries.

1.1 Overview of the Results

To illustrate our main results, consider the example of a planner who is connected to three intermediaries, who themselve are connected to two agents (see figure 1). The planner seeks to

[1] Thus, for instance, the Federal Emergency Management Agency may be more interested in allocating money to the Red Cross, which distribute a large percentage of their resources to helping domestic citizens affected by disasters, as opposed to UNICEF which helps children around the world.

[2] This factor in the quality of the charities is so important that all charities in the US are required by law to report the total percentage amount spent in their causes, as opposed to administration costs. For instance, the current indirect costs for the Red Cross and UNICEF are 9.7% and 4.74%, respectively. Multiple online websites exists that rank charities based on the indirect costs, among other metrics.

[3] The Red Cross publishes at the end of each year 'its activities in the field and at the headquarters during the coming year,' which allow donors to make informed decision on where the money will go. Earmarking is typically not allowed in such big charities, as 'experience shows that the more restrictive the earmarking policy (whereby donors require that their funds be allocated to a particular region, country, program, project or goods), the more limited the ICRC's operational flexibility, to the detriment of the people that the ICRC is trying to help.' https://www.icrc.org/en/support-us/where-does-your-money-go

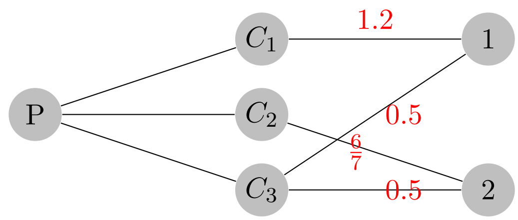

Figure 1: A network with three intermediaries and two agents.

transfer I units of money and has preferences over the allocation of the resource to the agents (y_1, y_2) given by a perfect complements utility function $u(y_1, y_2) = \min\{y_1, y_2\}$. Moreover, assume that intermediary C_1 can only transmit the resource to agent 1 and intermediary C_2 can only transmit the resource to agent 2. On the other hand, Intermediary C_3 can transmit the resource to agents 1 and 2, but it can only do so in equal proportions (we refer to this as the *quality of intermediation* of intermediary C_3, or *sharing-rates*). The qualities of intermediaries C_1 and C_2 are 1.2 and $\frac{6}{7}$, respectively. Thus, every unit of money sent to intermediary C_1 is increased by 20%, whereas every unit sent to intermediary C_2 is decreased by $\frac{1}{7}$.

In the absence of cost for using the intermediaries, the planner (P) can efficiently transmit the resource by using three potential groups. The planner P can use intermediaries C_1 and C_2, and transmit $\frac{5I}{12}$ and $\frac{7I}{12}$ via intermediaries C_1 and C_2 respectively. The final allocation to the agents is $(\frac{I}{2}, \frac{I}{2})$. Alternatively, the planner P can allocate all the resource to intermediary C_3, and the agents will also get the same maximal allocation $(\frac{I}{2}, \frac{I}{2})$. Moreover, the planner P can also use intermediaries C_1, C_2 and C_3 to transmit the resource efficiently by making a convex combination of the above.

Now, assume that the planner elicits costs from the intermediaries for using their links, and selects a group of intermediaries who are used at every vector of costs (c_1, c_2, c_3), where c_i is the fixed-cost that intermediary C_i reports. There are two types of SPNEs. The first equilibrium is the *efficient-SPNE*, where the cost profile is $(0, 0, 0)$. In this equilibrium, the planner fully transmits the resource to the agents without any cost paid to the intermediaries. This is an equilibrium because if an intermediary who is used by the planner increases its cost above zero, then the planner will not select it, as it can use another group of intermediaries to transmit the resource efficiently.

The second type of SPNE is the inefficient equilibrium, (c_1, c_2, c_3), where $c_1 \geq I$, $c_2 \geq I$ and $c_3 = I$. In this equilibrium, the planner pays intermediary C_3 an amount equal to I and transmits no resource to the agents. This is an equilibrium because neither intermediaries C_1 or C_2 can decrease their cost to undercut intermediary C_3. Intermediary C_3 has no incentive to decrease its cost because it is being selected.

Two results of the paper relate to the existence of an efficient-SPNE, where intermediaries charge zero cost at equilibrium. Theorem 1 shows that an Efficient-SPNE exists if and only if the intersection of the efficient groups (in our example above $\{C_1, C_2\}$, $\{C_3\}$ and $\{C_1, C_2, C_3\}$) is empty.

It is natural to also look for equilibrium where the group of agents who are not selected price at zero.[4] We call this equilibrium a *Robust-SPNE*. In our example, $(0, 0, 0)$ is the unique Robust equilibrium, since in the second type of SPNE intermediaries C_1 and C_2 charge a cost larger than I. Theorem 2 shows that if the planner has strongly-monotonic and homothetic preferences, then

[4]One can imagine that if intermediaries are not selected, then they will undercut their price trying to get selected, thus at the equilibrium it is natural to assume that the intermediaries who are not selected price at zero in equilibrium.

there is a unique Robust-SPNE if and only if the intersection of the efficient groups is empty.

We further explore the conditions for the existence of SPNEs, even though they may not be efficient. Theorem 3 provides three necessary and sufficient conditions for a cost c to be supported as a SPNEs when preferences are strongly monotone. The first condition requires that no intermediary should be in all the group-maximizing planner's utility under the cost c. The second condition requires that all the intermediary who belong to some utility maximizing group at cost c and charge positive cost should belong to the same utility maximization group. The third condition requires that the intermediaries who do not belong to any utility maximizing group at cost c should not change the indirect utility of the planner when they lower their cost to zero.

We also show the existence of a Robust SPNE when the intersection of the efficient groups is not empty. Proposition 1 shows conditions that guarantee the uniqueness of the robust SPNE. Proposition 2 provides conditions for the existence of a SPNE and uniqueness of the Robust SPNE from simple inequalities that depend on computable values from the utility function of the planner.

Comparative statics related to the addition of intermediaries are discussed in Corollaries 1 and 2. Corollary 1 shows that by replicating the intermediaries (to the points where there are at least two intermediaries of every type), or by adding one intermediary with 'high quality of links' will guarantee the existence of an efficient-SPNE and unique Robust-SPNE. Corollary 2 demonstrates that under some condition, adding one arbitrary intermediary will not make the planner worse off. In general, the effect of adding one intermediary is ambiguous due to the multiplicity of equilibria.

Finally, Theorem 4 studies the case when the planner may be able to determine the quality of the intermediaries (i.e., the planner is able to reallocate the proportions in which every intermediary distributes the resource to the agents). It shows that the necessary and sufficient conditions for the existence of an efficient SPNE is that every agent is connected to at least two intermediaries. At the same time, there are only two types of SPNE in which the intermediaries either charge the planner cost 0 or total cost I for using their links.

This is the first result in the literature of intermediation that works for a wide variety of preferences. In most of the results we only assume that the preferences of the planner are weakly-monotonic and homothetic over the allocation of resource to the agents.[5] All of the results in the paper work for the three canonical preferences in the literature: Cobb-Douglass, perfect complements and perfect substitutes.

1.2 Related Literature

The allocation of divisible resources in networks (see, Jackson[21] for the most comprehensive survey in networks) include Hougaard, et al.[15, 16, 17, 18, 19], Moulin[42], [43], Moulin et al.[44] and Juarez et al.[28, 29, 27]. However, we study the problem of transmitting a divisible good in networks with intermediaries, which not surprisingly, creates substantial differences in the equilibria, strategies and difficulty of the model. The most closely related paper is Moulin and Velez[46], which study the price of imperfect competition for the problem of spanning tree. The spanning tree model is similar with the resource transmitting in network with intermediary in the case of flexible sharing-rate (see Section 6). However, in our more general case, the problem of the planner depends on the various networks (including connections and quality of connections) and utility function, which result in more complex equilibriums than the ones discussed in Moulin and Velez[46].

There is also a large and growing literature in the transmission of indivisible goods and services with intermediaries. Many researches have studied the role of intermediation in agriculture, financial market and measure long supply chains. For instance, Li and Schurhoff[33] study network of intermediation in financial market about execution speed and trading cost. Antras and

[5] Although, we do assume strong monotonicity in some of the statements.

Costinot[2] and Antras and Chor[1] study trade with intermediation, they focus on the welfare effect of integration on the markets and the optimal allocation of ownership rights in a long supply chain.

At the same time, different trading mechanisms are discussed in the literature, including bilateral bargaining, price posted and second price auction. Manea[34] study dynamic game on bilateral bargaining in network with intermediation, Kotowski and Leister[30] study intermediary traders in network with auction mechanism to set prices, and analyze the welfare implications of stable and equilibrium networks. Blume et al.[5] and Gale and Kariv[14] study the effect intermediation in the market with price posted. Gale and Kariv[14] study the market with intermediaries and find the pricing behavior converges to competitive equilibrium in an experiment. Choi, Galeotti and Goyal[9] study pricing in complex structures of intermediation.

Competition and pricing of agents in networks has also been studied. For instance, Bloch[4] give a survey about targeting and pricing in social network. Bloch and Querou[3] study the monopoly pricing in social networks with consumer externalities. Campbell[7] study monopoly targeting and pricing with communication in the network of consumer. Our paper is related in a sense that we study the competition behavior among the intermediaries in a targeting problem.

Chawla and Niu[8] study the price of anarchy in Bertrand games on network. The price competition among the sellers resembles the price bidding of our mechanism. However, their model focus on the decision of buyers. Here, we study the problem from the planner's point of view. The network and utility function of the planner determines the competition among the intermediaries, rather than the standard Bertrand competition.

To our knowledge, this model is the first to consider the quality of intermediation, as well as to consider a wide variety of preferences of the planner.

1.3 Roadmap

Section 2 introduces the game of resource transmission in network with intermediaries. Section 3 studies the sufficient and necessary condition for efficient SPNE and uniqueness of robust SPNE. Section 4 analyzes the sufficient and necessary condition for SPNE in general case, and investigates the properties of robust SPNE. Section 5 presents the comparative statics results. Section 6 discusses extensions, interpretations, and applications of the model and provides concluding remarks.

2 The Model

Consider a planner who is interested in sending I units of a divisible resource to a group of M agents. We denote the agents by $1, 2, \ldots, M$ and the total group of agents by $A = \{1, \ldots, M\}$. The planner has preferences over the resource $y = (y_1, y_2, \ldots, y_M)$ sent to the agents. These preferences are denoted by the utility function $u : \mathbb{R}_+^M \to \mathbb{R}$ that is continuous and weakly-monotonic, see Figure 2.[6]

We consider the case where the planner is not directly connected to the agents, but instead, it is connected to a group of N intermediaries, denoted by $C_1, C_2, \ldots, C_N$, and let $C = \{C_1, C_2, \ldots, C_N\}$ be the subset of all intermediaries. We focus on the case where there are fixed links between the intermediaries and the agents. Every intermediary is connected to a group of agents, which is

[6]We impose further restriction on the preferences, such as strong monotonicity and homotheticity, in some of the results. All definitions about properties of the utility function are standard definitions from Mas-Colell, Whinston and Green[39] Chapter 3.

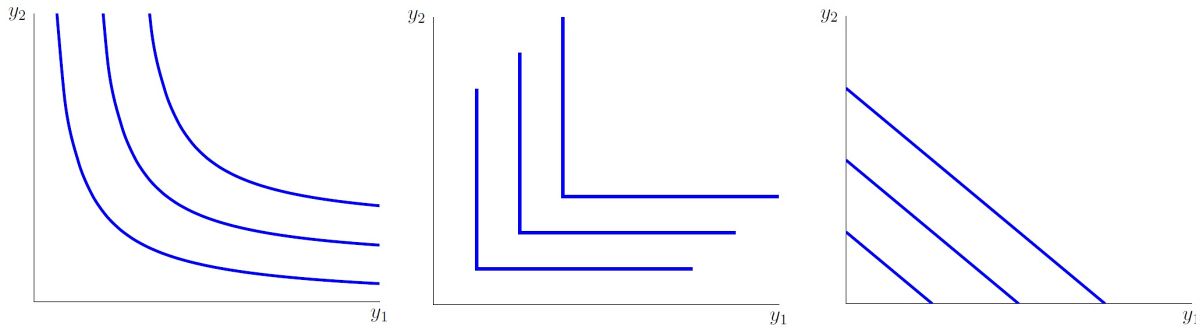

Figure 2: The model studied in this paper is very general and allows for basically any type of planner's preferences, including the three canonical cases: Cobb-Douglas, Perfect Complements and Perfect Substitutes.

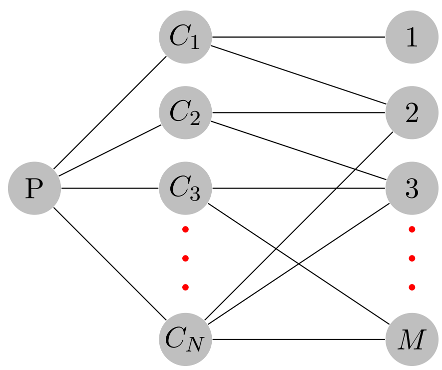

Figure 3: A generic graph for the model of intermediation.

denoted by the bipartite network $g \in \mathbb{M}^{N\times M}$, where $g_{nm} = 1$ if there is a link between intermediary n and agent m, and $g_{nm} = 0$ if there is no link between them. Figure 3 shows a generic model of intermediation.

Every intermediary can transmit resources to the agents that it is connected with some fixed quality, this is denoted by the *sharing-rate.* Let s_{nm} be the sharing-rate of intermediary C_n connected to agent m, where $0 \leq s_{nm}$ for each intermediary C_n. The matrix of sharing-rates is $S = (s_{11}, \ldots, s_{NM})_{N\times M}$. We assume that if there is no link between intermediary C_n and agent m, that is if $g_{nm} = 0$, then $s_{nm} = 0$. Thus, there is no transmission of resource between intermediary C_n and agent m when there is no connection between them. The sharing-rate distinguishes the way in which intermediaries transmit resources to agents per unit of money given.[7] Two intermediaries connected to the same group of agents might have different impacts in the agents, and thus one might be better aligned than the other to the planner's preferences.[8]

Let $\triangle^N(L) = \{x \in \mathbb{R}^N_+ | \sum_{i=1}^N x_i = L\}$ be the simplex in the space of intermediaries. Every

[7] When $\sum_{m=1}^M s_{nm} < 1$, we can interpret the intermediary as being inefficient. Such inefficiencies happen often in charities and universities, where every dollar spent is often decreased due to indirect cost, which serves to pay for administration. The case of $\sum_{m=1}^M s_{nm} > 1$ implies that a dollar spend in the intermediary expands, for instance when charities or universities offers matching funds from donors. Previous results in the transmission of resource in networks do not distinguish in the quality of the links, thus assume that the sharing-rate is equal or not fixed.

[8] For instance, as we will see below, the planner might prefer to select an intermediary who wastes more money but is better aligned with his preferences than an intermediary who does not wastes money but is less aligned to his preferences.

vector $x \in \triangle^N(L)$ represents a feasible assignment from the planner to the intermediaries when there are L units of resource available. Given a feasible assignment x, the final allocation $y_m(x)$ transmitted to the agent m satisfies $y_m(x) = \sum_{n=1}^N s_{nm} x_n$.

Definition 1 (Money Transmission Game with Intermediaries)
We study a two-stage perfect information game, where at the first stage, intermediaries choose simultaneously and independently a cost c for using their links. In the second stage and observing the costs c charged by the agents, the planner chooses a group of intermediaries who are used and a vector $x \in \mathbb{R}_+^N$ representing the amount allocated to these intermediaries.

- *The* ***strategy of intermediary*** *C_n is to set a fixed cost $c_n \geq 0$ that the planner has to pay to use its links. Let $c = (c_1, ..., c_N)$ be the vector of strategies by the intermediaries. The objective of each intermediary is to maximize the cost paid by the planner to use its links.*
- *The* ***strategy of the planner*** *is a function $b : \mathbb{R}_+^N \to \{0,1\}^N$ and $x : \mathbb{R}_+^N \to \mathbb{R}_+^N$ that allocates resources to intermediaries at every vector of costs c such that*

$$\sum_{n=1}^N x_n(c) = I - \sum_{n=1}^N b_n(c) c_n.$$

  Note that the planner only pays for the links used, therefore the total cost paid by planner is $\sum_{n=1}^N b_n(c) c_n$, and $I - \sum_{n=1}^N b_n(c) c_n$ units of resource are transmitted to the agents. Thus, $x(c) \in \triangle^N(I - \sum_{n=1}^N b_n(c) c_n)$. Furthermore, $b(c)$ and $x(c)$ are consistent in the sense that $x_n > 0$ only when $b_n = 1$, and $x_n = 0$ when $b_n = 0$, which means that only paid intermediaries can be used to transmit resource.
- *The* ***utility*** *$V^n(c,b)$* ***of intermediary*** *C_n is: $V^n(c,b) = c_n$ if $b_n(c) = 1$, and $V^n(c,b) = 0$ if $b_n(c) = 0$. That is, only the intermediaries selected might get positive utility equal to their proposed cost.*
- *The* ***utility of the planner*** *equals to $u(y(x(c)))$, which represents the utility at the final resource $y(x(c))$ transmitted to the agents.*

Let $b^*(c, I)$ be the vectors of intermediaries used to maximize utility and $x^*(c, I)$ be the subset of vectors of optimal allocations of the planner given his resource I and vector of cost c. The indirect utility function of the planner is $v(c, I) = u(y(\bar{x}))$ for some $\bar{x} \in x^*(c, I)$. The indirect utility function of the planner when not using the group of intermediaries P is denoted by $v_{-P}(c_{-P}, I)$.

Definition 2 (Subgame Perfect Nash Equilibrium)
The strategies from intermediaries $c \in \mathbb{R}_+^N$ and planner $b(c) : \mathbb{R}_+^N \to \{0,1\}^N$, $x(c) : \mathbb{R}_+^N \to \mathbb{R}_+^N$ are a ***Subgame Perfect Nash Equilibrium (SPNE)*** *if*

- *$V^n(c, b) \geq V^n(\tilde{c}_n, c_{-n}, b)$ for any $C_n \in C$ and $\tilde{c}_n \in \mathbb{R}_+$.*
- *$b(c) \in b^*(c, I)$, $x(c) \in x^*(c, I)$ for any c.*

The first best for the planner is to finding conditions under which there is no waste of resource used to pay the intermediaries. We capture this in the definition of Efficient-SPNE, where the final allocation chosen is as if the vector of zero costs is a SPNE.[9]

[9]Note that we focus in a utility function of the planner that cares only about the final resource transmitted to the agents and disregards the benefits of the intermediaries. This is distinct from the Pareto efficient allocations of other work in the literature that take into account the utility of the planner and the intermediaries.

Definition 3 (Efficient-SPNE)

- *Let $\bar{z} = 0$ be the vector of zero costs. An allocation of money to intermediaries x is efficient if $x \in x^*(\bar{z}, I)$.*
- *We say that (c, b, x) is an **Efficient-SPNE** if (c, b, x) is a SPNE and $x(c^*)$ is efficient.*

Note that an efficient SPNE requires that the allocation of money to the intermediaries is efficient, however not all intermediaries need to be pricing at zero cost. This will be seen in some of the examples below.

Example 1 (Substitute Network)
Consider the case of N identical intermediaries $C_1, \ldots, C_N$ connected to agents in an arbitrary network. Assume that the sharing-rates are equal among the intermediaries, that is, $s_{nm} = s_m$. The preferences of the planner are monotone. If there is only one intermediary who charges cost 0, then it has incentive to increase the cost slightly below the second lowest cost to be chosen and get higher utility. If no intermediary charges cost 0, every intermediary except the one with lowest cost has incentive to undercut. Then, the cost allocation c in a SPNE has at least two intermediaries who charge cost 0. It is easy to verify that every cost allocation such that $c_i = c_j = 0$, for some $C_i, C_j \in \{C_1, \ldots, C_N\}$ and $c_n \geq 0$, $\forall n \neq i, j$ is a SPNE. Moreover, there exists Efficient-SPNE in this model independent of the preferences of planner.

Example 2 (Linear Utility Function)
Consider a planner with utility function $u(y) = \sum_{m=1}^{M} \alpha_m y_m$, where α_m is the weight of the final resource allocated to agent m. Also, consider an arbitrary network. Given the sharing-rates $\{s_{nm}\}_{\{n \in N, m \in M\}}$, the marginal utility of resource allocated to intermediary C_n is constant and given by $MU_n = \sum_{m=1}^{M} \alpha_m s_{nm}$. This is independent of the allocation $x(c)$. Without loss of generality, we rename the intermediaries based in a non-increasing order of their marginal utility, that is $MU_1 \geq MU_2 \geq \cdots \geq MU_N$.

When $MU_1 = \cdots = MU_k > MU_{k+1}$ and $k \geq 2$, the planner is indifferent between allocating the resources to any of the intermediaries from $\{C_1, \ldots, C_k\}$ when their cost is zero. If only one intermediary from $\{C_1, \ldots, C_k\}$ has cost zero, then he can raise the price to slightly below the second highest intermediary and continue being chosen in a SPNE. If no intermediary from $\{C_1, \ldots, C_k\}$ has zero cost, then at most one of them will be chosen, and the ones who are not chosen have the incentive to decrease their cost. Therefore, a SPNE requires that at least two intermediaries from $\{C_1, \ldots, C_k\}$ have cost zero. It is easy to verify that every cost allocation such that $c_i = c_j = 0$, for some $C_i, C_j \in \{C_1, \ldots, C_k\}$ and $c_n \geq 0$, $\forall n \neq i, j$ is a SPNE. Thus, in this example there is an Efficient-SPNE.

When $MU_1 > MU_2$, the intermediary C_1 has some market power to price above zero and continue being chosen. In a SPNE, $c_2 = 0$ and $c_1 = I(1 - \frac{MU_2}{MU_1})$, $c_n \geq 0$, $\forall n \geq 3$ and intermediary C_1 is chosen to transmit $I - c_1$ units of resource. The planner's utility would be $I \cdot MU_2$, which is welfare equivalent to the utility given by allocating all resource to the intermediary with the second highest marginal utility when he prices at 0. In particular, there is no efficient-SPNE.

Example 3 (Symmetric Network)
Consider the network in Figure 4. Assume utility function is $u(y) = min\{y_1, y_2, y_3\}$, and the matrix of sharing-rates $S = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$.

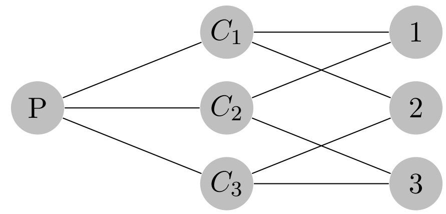

Figure 4: Network with three symmetric intermediaries

There is an equal-sharing between the intermediaries and the agents, every intermediary is connected to two agents and would always send the resource equally to the agents connected. The planner cares about the agent who is allocated the least resources. Note that, if the planner only uses two intermediaries, the optimal allocation is to transmit half resource through each intermediary. Thus, the agent connected to both intermediaries would get half of the resource and each of the other two agents would get one-quarter of the resource transmitted. In this case, the resource cannot be allocated equally to three agents and results in waste of resource and inefficiency. Thus, the only way to achieve efficiency is to use all three intermediaries, the planner is able to transmit the resource equally to the agents.

Every intermediary is necessary to be used to achieve the efficiency in transmitting certain amount of resource, so it has some market power to charge positive cost in equilibrium. There is a symmetric equilibrium where every intermediary charges $\frac{I}{6}$, that is $c = (\frac{I}{6}, \frac{I}{6}, \frac{I}{6})$, the planner would use all the intermediaries $b(c) = (1, 1, 1)$ with allocation of resource $x(c) = (\frac{I}{6}, \frac{I}{6}, \frac{I}{6})$, and it is indifferent with using any two intermediaries. If C_1 increase its cost, it would results in lower utility of using all intermediaries, while the utility of using the $\{C_2, C_3\}$ remains the same, then the planner would choose $\{C_2, C_3\}$, and C_1 lose the payment. The intermediaries also have no incentive to lower their cost, since they are already being selected. Thus, this cost allocation is an equilibrium.

There is another equilibrium cost allocation, which results when every intermediary charges a total resource I, that is $c = (I, I, I)$, and the planner pays one of the intermediaries (say, intermediary C_1, $b(c) = (1, 0, 0)$) all the resource without transmitting anything, which means $x = (0, 0, 0)$. In this equilibrium, there is no incentive for C_1 to deviate, since it gets all the resource. For C_2 or C_3, even if one decreases the cost, the planner cannot get positive utility because one intermediary is not connected to all the agents and at least one agent would receive 0 resource. Thus paying all resource to C_1 is still the best strategy for planner. This very inefficient SPNE exists because intermediaries C_2 and C_3 cannot cooperate by lowering their cost simultaneously.

3 Efficient-SPNE: Existence and Uniqueness

Definition 4 (Utility Maximization Group)

- *Given the vector of costs $c \in \mathbb{R}_+^N$, we say that $P \subset C$ is a* ***utility maximization group*** *if there exists $\bar{x} \in x^*(c, I)$ such that $C_n \in P \Leftrightarrow \bar{x}_n > 0$.*

- *An* ***efficient group*** *P^e is a utility maximization group under cost $c = 0$. Let $P_1^e, \ldots, P_J^e$ be the efficient groups.*

One necessary condition for the existence of SPNE, is that the utility maximization group must be chosen by the planner at any c. In order to find an Efficient-SPNE, one of the efficient groups

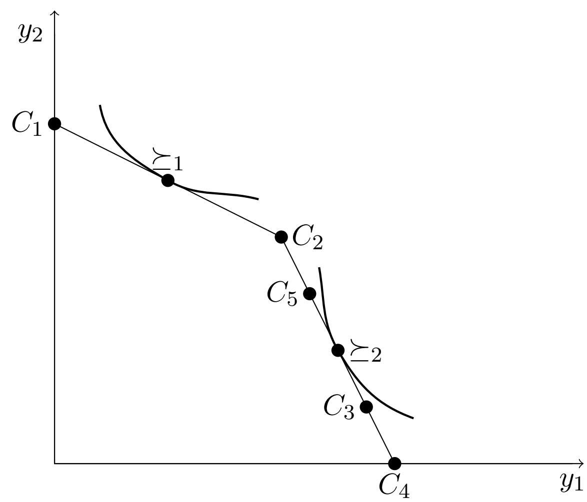

Figure 5: Efficient groups for preferences $\succeq_1$ and $\succeq_2$

must be chosen.

Example 4 (Utility maximization groups for examples 1, 2 and 3.)

- *In example 1, the efficient groups include all subset of $\{C_1, \ldots, C_N\}$. For example, $\{C_1\}$, $\{C_2\}$,..., $\{C_N\}$ are all efficient groups.*
- *In example 2, assume $\{C_1, \ldots, C_k\}$ are the subset of intermediaries with marginal utility $MU^n = MU^1$, $\forall 1 \leq n \leq k$. The efficient groups include all subset of $\{C_1, \ldots, C_k\}$.*
- *In example 3, the only efficient group is $\{C_1, C_2, C_3\}$.*

Example 5 (Utility maximization groups for different preferences)
In this example we illustrate that the utility maximization group(s) are very dependent on the preferences of the planner. We also illustrate the graphic computation of efficient groups for arbitrary preferences. Consider the case in figure 5. There are 2 agents, and intermediaries $\{C_1, \ldots, C_5\}$. The dots in the graph labeled with the intermediaries' name represent the final allocation of resource transmitted to the agents if such intermediary is used to transmit I units of the resource. Every point in the line between two intermediaries can be achieved by making a convex combination of the resource to the intermediaries. Therefore, the 'Pareto-frontier' of the dots represent all potentially efficient allocations.

If the preferences are given by $\succeq_1$, using C_1 and C_2 will achieve the allocation of maximal utility, the efficient group is therefore $\{C_1, C_2\}$.

Alternatively, if the preferences are given by $\succeq_2$, the allocation that maximizes the utility can be achieved with groups $\{C_2, C_3\}$, $\{C_2, C_4\}$, $\{C_3, C_5\}$, $\{C_4, C_5\}$ (and their respective unions).

Example 6 (Efficient SPNE and group selection)
Consider the network in Figure 6. Assume that the utility function is $u(y) = min\{y_1, y_2, y_3, y_4\}$

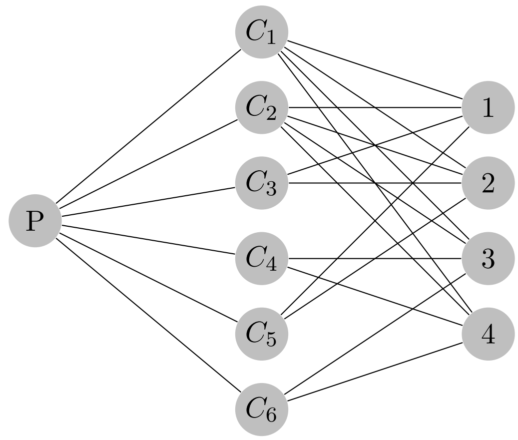

Figure 6: Network with Multiple Equilibria

and the matrix of sharing-rates is $S = \begin{bmatrix} \frac{1}{6} & \frac{1}{3} & \frac{1}{6} & \frac{1}{3} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{6} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$. *The efficient groups include* $P_1^e = \{C_3, C_4\}$, $P_1^e = \{C_5, C_6\}$. *But there is an inefficient equilibrium* $c = (0, 0, c_3, c_4, c_5, c_6)$, $c_3, c_4, c_5, c_6 \geq \frac{I}{3}$, *in which the planner chooses* $b(c) = (1, 0, 0, 0, 0, 0)$, $x(c) = (I, 0, 0, 0, 0, 0)$ *and provides a final utility* $u(y) = \frac{I}{6}$. *In this equilibrium,* C_1 *and* C_2 *compete and have no incentive to raise their cost,* C_3, C_4, C_5, C_6 *will not be selected if they decrease their cost by themselves, thus they have no incentive to lower their cost. At the same time, there is an efficient equilibrium* $c = (c_1, c_2, 0, 0, 0, 0)$, $c_1, c_2 \geq 0$, $b(c) = (0, 0, 1, 1, 0, 0)$, $x(c) = (0, 0, \frac{I}{2}, \frac{I}{2}, 0, 0)$ *that provides a final utility* $u(y) = \frac{I}{4}$.

From this example, we can see that in some SPNE, the planner can pay a zero cost for using some intermediaries, but the result of resource allocation is not efficient, $\frac{I}{6} < \frac{I}{4}$. *Therefore, in order to find an Efficient SPNE, the efficient group must be pricing at zero cost.*

Theorem 1 (Existence of efficient SPNE)
Assume the efficient groups are P_1^e, P_2^e, ..., P_J^e. *There is an Efficient SPNE if and only if* $\bigcap_{j=1}^{J} P_j^e = \emptyset$.

From this result, the existence of an efficient equilibrium implies that there is no coalition of intermediaries who belong to all the efficient groups. The intuition is that, if a coalition of intermediaries belong to this intersection, then these intermediaries will have sufficient market power to price above zero, thus creating an inefficient equilibrium. The extreme case occurs in the traditional Bertrand competition model, where symmetric producers compete for a price and the only SPNE leads to zero cost when the marginal cost of production is 0, as in Example 1 above. However, when intermediaries have different marginal cost of production in the Bertrand competition, a SPNE where intermediaries price above zero is possible.

Proof.

$\Leftarrow$) If $\bigcap_{j=1}^{J} P_j^e = \emptyset$, $\forall n$, given $c_{-n} = 0$, when C_n choose $c_n > 0$, there exists group P_j^e with $C_n \notin P_j^e$, since the intermediaries in P_j^e charge cost 0. The maximal utility of planner by using a

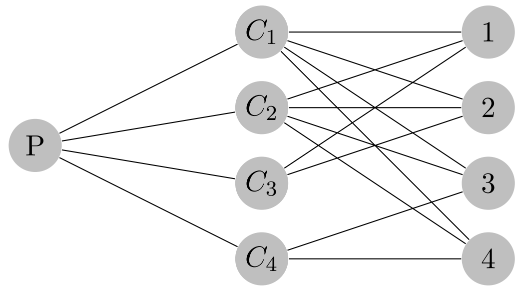

Figure 7: Network with Complements Intermediaries

group with C_n would not be higher than the efficient group P_j^e. C_n deviate from $c_n = 0$ to $c_n > 0$ would not increase its benefit. Thus, $c_n = 0$, $\forall n$ is SPNE.

$\Rightarrow$) If $\bigcap_{j=1}^{J} P_j^e \neq \emptyset$, we show that $c_n = 0\ \forall n$ is not SPNE. Indeed, pick $C_n \in \bigcap_{j=1}^{J} P_j^e$. Without using intermediary C_n, given the cost of other intermediaries c_{-n}, assume the maximal utility is $v_{-n}(c_{-n}, I)$. Since C_n is used in all efficient groups P_j^e, then $v(c, I) > v_{-n}(c_{-n}, I)$. So, there exists ϵ small enough such that C_n increases the cost c_n to ϵ, $c_n' = \epsilon$, $c' = (c_1, ..., c_{n-1}, c_n', c_{n+1}, ..., c_N)$, s.t $v(c, I) > v(c', I) > v_{-n}(c_{-n}, I)$. Thus, C_n would deviate and get positive profit. Therefore, $c = 0$ is a SPNE only if $\bigcap_{j=1}^{J} P_j^e = \emptyset$. ■

3.1 Multiplicity of SPNE

Example 7 (Multiple Equilibria)
Consider the network in Figure 7. Assume utility function is $u(y) = min\{y_1, y_2, y_3, y_4\}$, the matrix of sharing-rates $S = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{6} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$. There are two efficient group $P_1^e = \{C_1\}$, $P_2^e = \{C_3, C_4\}$. But there is inefficient equilibrium $c = (\frac{I}{3}, 0, c_3, c_4)$, $c_3 \geq \frac{I}{3}$, $c_4 \geq \frac{I}{3}$, with planner choose $b(c) = (1, 0, 0, 0)$, $x(c) = (\frac{2I}{3}, 0, 0, 0)$, $u(y) = \frac{I}{6}$ which guarantees C_1 has no incentive to decrease cost, and C_3, C_4 have no benefit deviating to charge lower cost independently. At the same time, there is efficient equilibrium $c = (0, c_2, 0, 0)$, $c_2 \geq 0$, $b(c) = (1, 0, 1, 1)$, $x(c) = (\frac{I}{3}, 0, \frac{I}{3}, \frac{I}{3})$, $u(y) = \frac{I}{4}$.

From this example, we can see the effect of adding an intermediary on the SPNEs. When there are only intermediaries $\{C_1, C_3, C_4\}$, there is a SPNEs with $c = (0, 0, 0)$, $u(y) = \frac{I}{4}$ and $c = (I, c_3, c_4)$, with $c_3, c_4 \geq I$. In this case, the planner pays C_1 all the resource I, $b(c) = (1, 0, 0)$, and has utility 0. After adding C_2, cost profiles $c = (0, c_2, 0, 0)$ and $c = (\frac{I}{3}, 0, c_3, c_4)$ are SPNEs. Both cases have efficient-SPNEs, while the inefficient SPNEs change from $c = (I, c_3, c_4)$ to $c = (\frac{I}{3}, 0, c_3, c_4)$, total cost paid to intermediaries decrease after adding intermediary.

In section 5 we discuss more about the benefits of adding intermediaries.

Example 8 (Multiple Equilibria)
Consider the network in Figure 8, replicate the intermediaries $\{C_1, C_3, C_4\}$ with the same links and

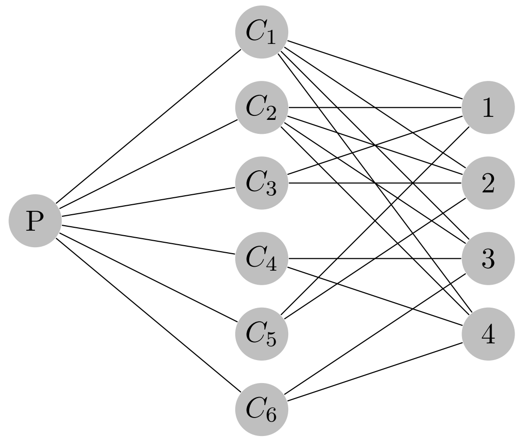

Figure 8: Network with Multiple Equilibria

sharing-rates to generate the network. The matrix of sharing-rates is $S = \begin{bmatrix} z & \frac{1-2z}{2} & z & \frac{1-2z}{2} \\ z & \frac{1-2z}{2} & z & \frac{1-2z}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$.

Assume $z \leq \frac{1}{4}$, *utility function is* $u(y) = min\{y_1, y_2, y_3, y_4\}$. *There are efficient groups* $P_1^e = \{C_3, C_4\}$, $P_2^e = \{C_5, C_6\}$, $P_3^e = \{C_3, C_6\}$, $P_4^e = \{C_4, C_5\}$.

Using the same method to find the SPNEs of this problem. In SPNE, $c = (0, 0, c_3, c_4, c_5, c_6)$ *with* $c_3, c_4, c_5, c_6 \geq I - 4zI$, $x(c) = (\frac{I}{2}, \frac{I}{2}, 0, 0, 0, 0)$, $u(y) = zI$ *or* $c = (c_1, c_2, 0, 0, 0, 0)$ *with* $c_1, c_2 \geq 0$, $b(c) = (0, 0, 1, 1, 1, 1)$, $x(c) = (0, 0, \frac{I}{4}, \frac{I}{4}, \frac{I}{4}, \frac{I}{4})$, $u(y) = \frac{I}{4}$.

Comparing to the case of intermediaries $\{C_1, C_3, C_4\}$, *the efficient group is* $P_1^e = \{C_3, C_4\}$. *In a SPNE,* $c = (I, I, I)$ *with* $b(c) = (1, 0, 0)$, $x(c) = (0, 0, 0)$, *planner pays* I *to* C_1, $u(y) = 0$ *or* $c = (0, c_2, c_3)$, $c_2 + c_3 = (1 - 4z)I$ *with* $b(c) = (0, 1, 1)$, $x(c) = (0, 2zI, 2zI)$, $u(y) = zI$.

This example shows replicating some intermediaries might improve the planner's utility in both worst equilibrium from 0 *to* zI, *and also the best equilibrium from* zI *to* $\frac{I}{4}$. *When* $z \to 0$, *the best equilibrium improves from* 0 *to* $\frac{I}{4}$, *and the worst equilibrium has little improvement. When* $z \to \frac{1}{4}$, *the worst equilibrium improves from* 0 *to* $\frac{I}{4}$, *and the best equilibrium improves little.*

More general results about the existence of an efficient equilibrium in 'replicated' networks is discussed in Corollary 2.

Example 9 (Price of Anarchy and Price of Stability)
Given the multiple SPNE in the game, two measures of efficiency have been used in the literature. The Price of Anarchy (POA) is defined as the ratio of the worst equilibrium over the optimal allocation, $POA = \frac{Worst\ SPNE}{Optimal}$. *On the other hand, the Price of Stability (POS) is defined as the ratio of the best equilibrium over the optimal allocation,* $POS = \frac{Best\ SPNE}{Optimal}$.

In example 7, the planner's utility of worst SPNE $\underline{u} = \frac{I}{6}$, *utility of best SPNE* $\bar{u} = \frac{I}{4}$, *utility of optimal allocation* $u_O = \frac{I}{4}$. $POS = 1$, *and* $POA = \frac{2}{3}$.

In example 8, the planner's utility of worst SPNE $\underline{u} = zI$, *utility of best SPNE* $\bar{u} = \frac{I}{4}$, *utility of optimal allocation* $u_O = \frac{I}{4}$. $POS = 1$, *and* $POA = 4z$. *When* $z \to 0$, $POA \to 0$, *which is as if the planner was not transmitting any resource to the agents.*

In particular, from example 8, the POA can be as close as zero when the POS can be as close at 1, and thus the efficiency varies a lot in different SPNE. Thus, in the extreme case, the SPNEs can simultaneously reach an optimal equilibrium and a fully inefficient equilibrium. The inefficient result is caused by the failure of coordination among group of intermediaries to lower their costs at the same time.

3.2 Efficient Robust-SPNE

In examples above we saw that multiplicity of equilibria is possible. Moreover, in example 9 we saw that this multiplicity implies that the utility of the agents can vary greatly from the optimal allocation at equilibrium (when $POS = 1$) to almost no utility for the planner (when $POA \approx 0$). However, we can argue that some of the equilibria might not be as likely to occur because there are groups of intermediaries who are not used and offer their links at a positive cost. These intermediaries who are not used always have the incentive to undercut their cost in hopes to be chosen.

In this section we look at a robustness of SPNE, where intermediaries who are not being used by the planner cannot jointly decrease their cost and affect the equilibrium. Formally, a SPNE is a robust equilibria when the intermediaries who are not used by the planner charge cost zero.

Definition 5 (Robust Equilibrium)
The Subgame Perfect Nash Equilibrium $(c, b(c), x(c))$ is robust if intermediaries who are not used by the planner charge zero cost. That is, the SPNE $(c, b(c), x(c))$ is robust whenever $b_i(c) = 0$ implies $c_i = 0$.

Example 10 (Robust SPNE)
In example 7, $c = (\frac{I}{3}, 0, c_3, c_4)$, $c_3 \geq \frac{I}{3}$, $c_4 \geq \frac{I}{3}$ and the planner choosing $b(c) = (1, 0, 0, 0)$, $x(c) = (\frac{2I}{3}, 0, 0, 0)$ are SPNEs. Note C_3, C_4 are not used by planner while charging positive cost, therefore these SPNE are not robust.

On the other hand, $c = (0, c_2, 0, 0)$, $c_2 \geq 0$, and the planner choosing $b(c) = (1, 0, 1, 1)$, $x(c) = (\frac{I}{3}, 0, \frac{I}{3}, \frac{I}{3})$ are SPNEs. In this case, C_2 is not used by planner, thus when C_2 charges 0 we get a robust SPNE. So only $c = (0, 0, 0, 0)$ is a cost allocation for a robust SPNE.

In example 8, SPNEs are $c = (0, 0, c_3, c_4, c_5, c_6)$, $c_3, c_4, c_5, c_6 \geq I - 4zI$ and the planner choosing $b(c) = (1, 0, 0, 0, 0, 0)$, $x(c) = (I, 0, 0, 0, 0, 0)$ are SPNEs. C_3, C_4, C_5, C_6 are not used by planner while charging positive cost, thus there are not robust SPNE.

$c = (c_1, c_2, 0, 0, 0, 0)$ with $c_1, c_2 \geq 0$ and planner choosing $b(c) = (0, 0, 1, 1, 1, 1)$, $x(c) = (0, 0, \frac{I}{4}, \frac{I}{4}, \frac{I}{4}, \frac{I}{4})$ are SPNEs. C_1 and C_2 will not be used by planner, thus they charge zero cost in a robust SPNE. So only $c = (0, 0, 0, 0, 0, 0)$ is a cost allocation for a robust SPNE.

Theorem 2 (Efficient Robust-SPNE)

1. *If $c_n = 0$ for all $C_n \in C$ is a Robust-SPNE then $\bigcap_{j=1}^{J} P_j^e = \emptyset$.*

2. *Assume that the planner has strongly-monotonic and homothetic preferences*[10]*. If $\bigcap_{j=1}^{J} P_j^e = \emptyset$ then $c_n = 0$ for all $C_n \in C$ is the unique Robust-SPNE.*

[10] Preferences are *homothetic* if and only if there exists a utility function such that $u(\lambda y) = \lambda u(y)$ for any $\lambda > 0$ and $y \in \mathbb{R}^N$. Note than if preferences are homothetic and $u(Q, I)$ is the maximal utility of planner using certain group Q of intermediaries by allocating I resource, then $u(Q, I) = I \cdot u(Q, 1)$ is the maximal utility of distributing 1 unit of resource with group Q. Let $u(Q) = u(Q, 1)$. If utility function $u(y) = min\{y_1, \ldots, y_M\}$, and $v(0, I) > 0$, the conclusion holds although it is not strongly monotonic.

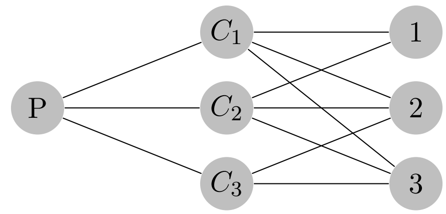

Figure 9: Network with a unique inefficient SPNE

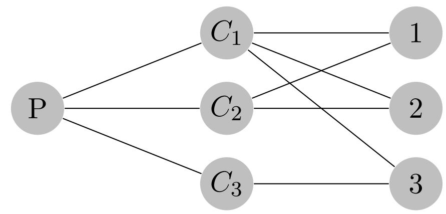

Figure 10: Network with multiple inefficient SPNE

This result is similar with Theorem 1. Under the condition of strongly monotonic and homothetic preferences and when no group of intermediaries belong to all the efficient coalitions, there exists a unique efficient robust equilibrium. The robust SPNE refines the SPNE, as it eliminates the equilibria where intermediaries cannot coordinately lower the cost to be used by planner. This simple robustness lead to a uniqueness of robust-SPNE under most preferences.

Proof. Part 1 is a trivial consequence of Theorem 1 because $(0, 0, \ldots, 0)$ is a Robust-SPNE.

In order to prove part 2, assume there is a robust SPNE with $c \neq 0$. From the definition of Robust SPNE, $\forall c_n > 0$, intermediary C_n is used in the utility maximization group of intermediaries. From part (2) of Theorem 3, there exists a utility maximization group P_1, such that all the intermediaries with positive cost are in P_1, that is, $\forall c_n > 0$, $C_n \in P_1$. The maximal utility for planner to use group P_1 is $(I - \sum_{c_n>0} c_n)u(P_1)$. The efficient groups are P_1^e, $\ldots$, P_J^e, assume $\overline{u} = u(P_1^e)$. Since $\bigcap_{j=1}^J P_j^e = \emptyset$, for intermediary C_k with $c_k > 0$, $\exists P_j^e$, $C_k \notin P_j^e$. Then the maximal utility for planner to use P_j^e is $(I - \sum_{C_n \in P_j^e} c_n)u(P_j^e) = (I - \sum_{C_n \in P_j^e} c_n)\overline{u}$. $I - \sum_{C_n \in P_j^e} c_n > I - \sum_{c_n>0} c_n$, and $\overline{u} \geq u(P_1)$, we have $(I - \sum_{c_n>0} c_n)u(P_1) < (I - \sum_{C_n \in P_j^e} c_n)\overline{u}$, it means group P_j^e results in strictly higher utility than P_1, which contradicts that P_1 is a maximal utility group. Thus, $c = 0$ is the unique robust SPNE. ■

4 Inefficient SPNE

From Theorem 1 we saw that an equilibrium with zero cost will happen if and only if $\bigcap_{j=1}^J P_j^e = \emptyset$. We see below that whenever $\bigcap_{j=1}^J P_j^e \neq \emptyset$ inefficient equilibria may exist. In aggregate, the intermediaries in the intersection may make a positive profit in equilibrium (there may be intermediaries in the intersection that will charge positive cost and will be selected), whereas the intermediaries who do not belong to the intersection may or may not charge positive cost and make profit at equilibrium.

Example 11 (Unique inefficient equilibrium)
Consider the network in Figure 9. Assume utility function is $u(y) = min\{y_1, y_2, y_3\}$, and the

matrix of sharing-rates $S = \begin{bmatrix} \frac{2}{3} & \frac{1}{6} & \frac{1}{6} \\ \frac{5}{6} & \frac{1}{12} & \frac{1}{12} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$. *The efficient group are* $P_1^e = \{C_1, C_3\}$, $P_2^e = \{C_2, C_3\}$, $P_3^e = \{C_1, C_2, C_3\}$. *Thus,* $\bigcap_{j=1}^3 P_j^e = \{C_3\}$. *The intersection of efficient groups* $\bigcap_{j=1}^3 P_j^e$ *is not empty,* C_3 *has market power to charge positive cost, while there is competition between* C_1 *and* C_2*. If* $c > 0$*, then a group that includes both* C_1 *and* C_2 *cannot be a utility maximization group, since the planner needs to pay* $c_1 + c_2 + c_3$ *using* P_3^e*, while only pay* $c_1 + c_3$ *using* P_1^e*. So* C_1 *or* C_2 *would be used. If* C_1 *is used, then* $c_1 < I$*, because when* $c_1 = I$*, any* $c_2 < I$ *would be used by planner. If* $P_1^e = \{C_1, C_3\}$ *is not utility maximization group, then it means* C_1 *is chosen,* C_3 *is not paid and has incentive to deviate to* $\frac{I-c_1}{2} > c_3 > 0$ *which guarantees* $\frac{I-c_1}{6} < \frac{I-c_1-c_3}{3}$ *and* C_3 *would get* c_3*, so* P_1^e *is a utility maximization group. Then if* C_2 *charges cost* $c_2 < c_1$*,* P_2^e *becomes utility maximization group,* C_2 *will deviate and get higher utility. The case of* C_2 *being used results in the same problem,* $c > 0$ *is not cost allocation in SPNE, then* $c_1 = 0$ *or* $c_2 = 0$*. If* $c_2 = 0$*,* $c_1 > 0$*, the only possible utility maximization groups are* P_2^e *and* C_1*, but either* P_2^e *or* C_1 *is chosen by planner,* C_1 *or* C_3 *has incentive to lower its cost. If* $c_1 = 0$*,* $c_2 > 0$*, then the only possible utility maximization groups are* P_1^e *and* C_1*,* C_1 *is always being chosen thus it could get positive benefit increasing the cost. From the analysis above, in SPNE,* $c_1 = c_2 = 0$*. It is not hard to find* $c = (0, 0, \frac{I}{2})$*,* $b(c) = (1, 0, 1)$*,* $x(c) = (\frac{I}{4}, 0, \frac{I}{4})$ *is SPNE, the planner will transmit the resource with intermediary* C_1 *and* C_3*. This is the unique cost* c *in a SPNE.*

This example shows when there is one intermediary in the intersection, then there is equilibrium it charges a positive cost such that the planner is indifferent between paying the cost and using it and using other intermediaries. In other words, the intermediary could charge the cost equal to its marginal contribution to the utility of planner, $v(0, I - c_1) = v_{-1}(0, I)$, assume C_1 is in the intersection.

Example 12 (Multiple inefficient equilibria)
Consider the network in Figure 10. Assume utility function is $u(y) = min\{y_1, y_2, y_3\}$*, and the matrix of sharing-rates* $S = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$. *The efficient group is* $P_1^e = \{C_2, C_3\}$. $\bigcap P_j^e = \{C_2, C_3\}$. *There are two types of SPNEs, both of them are inefficient. The first type is such that* $c = (0, c_2, c_3)$ *with* $c_2 + c_3 = \frac{I}{4}$*,* $b(c) = (0, 1, 1)$*,* $x = (0, \frac{I}{2}, \frac{I}{4})$ *and the final utility of the planner equals to* $u = \frac{I}{4}$*. The second type of SPNE is* $c = (c_1, c_2, c_3)$*, with* $c_1 = I$*,* $c_2, c_3 \geq I$ *and* $b(c) = (1, 0, 0)$*,* $x(c) = (0, 0, 0)$*, in which case* C_1 *gets the full resource and the planner does not transmit anything to the agents, so the final utility of the planner is 0. In this case, intermediary* C_2 *and* C_3 *are complements, so they cannot unilaterally deviate to the more efficient SPNE.*

In this example, there are two intermediaries in the intersection, but they need to be used at the same time. The complementarity between these two intermediaries result the sum of costs are constant, which is equal to their marginal contribution in equilibria.

4.1 Properties of SPNE

Theorem 3 (Conditions for SPNE)
Consider a cost and allocation $(c, b(c), x(c))$ *that is a SPNE. Assume that the utility maximization groups at cost* c *are* $P_1(c), ..., P_K(c)$*. Then,*

1. $\bigcap_{k=1}^K P_k(c) = \emptyset$.

2. *If the preferences are strongly monotone, then there exists a utility maximization group, without loss of generality, assume it is* $P_1(c)$, *s.t.* $\forall C_n \in \bigcup_{k=1}^K P_k(c) \backslash P_1(c)$, $c_n = 0$.[11]

3. $\forall C_j \in C \backslash \bigcup_{k=1}^K P_k(c)$, *it would not increase the planner's utility even if its cost decreases to* 0. *That is, for* $c'_j = 0$ *and* $c' = (c_{-j}, c'_j)$, *we have that* $v(c, I) = v(c', I)$.

Conversely, if there exists a cost c that satisfies conditions 1-3, then it is SPNE.

Intuitively, condition 1 means that in a SPNE, the intermediaries chosen cannot increase the cost to get a higher benefit. As a result, there will be at least two utility maximization groups for any SPNE. Condition 2 means that in a SPNE, the intermediaries in some utility maximization group with positive cost have no incentive to decrease their cost. Condition 3 means that the intermediaries not used in the utility maximization group would not be used even if they decrease the cost to 0.

Proof.

1. Suppose $\bigcap_{k=1}^K P_k(c) \neq \emptyset$, there exists $C_n \in \bigcap_{k=1}^K P_k(c)$. Without using intermediary n, given the cost of other intermediaries c_{-n}, consider the planner's utility maximization problem, $max_{x_{-n}} u(y)$, s.t $\sum x_{-n} + b_{-n} c_{-n} \leq I$. Assume the maximal utility is $v_{-n}(c_{-n}, I)$. Since C_n is in every utility maximization group, the utility maximization group in $C \backslash C_n$ would achieve lower utility given cost c_{-n}, then, $v_{-n}(c_{-n}, I) < v(c, I)$. The indirect utility function is continuous, $\exists \epsilon$ small enough, such that intermediary n increases its cost by ϵ, $c'_n = c_n + \epsilon$, $c' = (c_{-n}, c'_n)$, s.t $v(c, I) > v(c', I) > v_{-n}(c_{-n}, I)$. Intermediary n would still be used by the planner after increasing cost by ϵ, there is incentive for C_n to deviate, the c cannot be SPNE. So $\bigcap_{k=1}^K P_k(c) = \emptyset$.

2. Assume we cannot find such a utility maximization group $P(c)$ that includes all positive cost intermediaries in $\bigcup_{k=1}^K P_k(c)$. Assume the planner choose group $P_1(c)$ to pay and allocate the resource with intermediaries in $P_1(c)$. Then, there exists $C_l \in \bigcup_{k=1}^K P_k(c) \backslash P_1(c)$ with $c_l > 0$. Take intermediary C_l and lower the cost by a small amount $\epsilon > 0$, $c'_l = c_l - \epsilon > 0$. Given the strongly monotonicity of preferences, using a group $P_k(c)$ that includes C_l would achieve strictly higher utility for the planner, and the groups without C_l would give the same utility for the planner. Thus, C_l would be used by the planner and have strictly higher profit $c_l - \epsilon$ rather than 0, so C_l has incentive to deviate, and cost c cannot be an equilibrium. Therefore, there exists a utility maximization group $P_1(c)$ that contains all intermediaries charging positive cost in $\bigcup_{k=1}^K P_k(c)$.

If $u(y) = min\{y_1, \ldots, y_M\}$, and $v(c, I) > 0$, it means the utility maximization group $P_k(c)$ have link to all the agents $\{1, \ldots, M\}$. When intermediary C_l lower the cost by small amount $\epsilon > 0$, the maximal utility will increase strictly. The proof works for this case, though the preferences are not strongly monotone.

3. For C_i not to be in any utility maximization group $P_k(c)$, it will not be used by the planner. If it lowers its cost and improve the maximal utility achieved by the planner, it has incentive to lower its cost and get paid. To make sure this case will not happen in a SPNE, it requires that even when the cost decreases to 0, the maximal utility of the planner would not increase. Thus, if $c'_j = 0$, $c' = (c_{-j}, c'_j)$, then $v(c, I) = v(c', I)$.

From the proof above, if c is a SPNE cost allocation, then 1, 2, 3 hold. On the other hand, to prove the converse, consider c satisfying conditions 1, 2, 3 and assume that the planner chooses the group $P_1(c)$ in condition 2, paying all intermediaries in $\bigcup_{k=1}^K P_k(c)$ with positive cost. Since $P_1(c)$ is a utility maximization group, the planner will achieve the maximal utility under c. For intermediary C_j, if $C_j \in C \backslash \bigcup_{k=1}^K P_k(c)$, from condition 3, it has no incentive to deviate. If $C_j \in \bigcup_{k=1}^K P_k(c)$,

[11] Even if preferences are perfect complements, that is $u(y) = min\{y_1, \ldots, y_M\}$, and $v(c, I) > 0$, this conclusion holds.

intermediary C_j has no incentive to increase its cost by condition 1, because the planner will use a different efficient group of intermediaries if its cost is higher. At the same time, this intermediary will not decrease its cost by condition 2, because the planner pays for its cost and lowering it will give the intermediary smaller profit. Thus, any cost c satisfying conditions 1, 2, 3 could be a cost allocation in SPNE. ■

Theorem above provides necessary and sufficient conditions for some cost c to be SPNE. However, in general there is no guarantee that such cost c will exist. We explore this in the next two propositions.

4.2 Uniqueness of Robust SPNE

In this section we answer the questions of existence of robust SPNE in the case of $\bigcap_{j=1}^{J} P_j^e \neq \emptyset$. In the following proposition, we will show that only the intermediaries in the intersection $\bigcap_{j=1}^{J} P_j^e$ might charge positive cost in a robust SPNE. In order to do so, first we define the perfect complementarity between the intermediaries in $\bigcap_{j=1}^{J} P_j^e$.

Definition 6 (Perfect Complements in Efficient Group)
Consider the intermediaries in $\bigcap_{j=1}^{J} P_j^e = \{C_1, \ldots, C_k\}$. We say that the agents in the group $\bar{C} \subset \{C_1, \ldots, C_k\}$ are perfect complements if $v_{-n}(0, I) = v_{-\bar{C}}(0, I)$, $\forall n \in \bar{C}$.

Proposition 1 (Uniqueness of Robust SPNE)
Assume preferences are homothetic and strongly monotonic. The following conditions are satisfied:

1. *At any Robust SPNE, only the intermediaries in the intersection $P = \bigcap_{j=1}^{J} P_j^e$ might charge positive cost.*
2. *When $|\bigcap_{j=1}^{J} P_j^e| \leq 1$, there is unique Robust SPNE.*
3. *When $\bigcap_{j=1}^{J} P_j^e = \{C_1, C_2\}$, there exists Robust SPNE. If $\frac{1}{\bar{u}_{12}} + \frac{1}{\bar{u}} \geq \frac{1}{\bar{u}_1} + \frac{1}{\bar{u}_2}$* [12]*, there is unique Robust SPNE. If $\frac{1}{\bar{u}_{12}} + \frac{1}{\bar{u}} < \frac{1}{\bar{u}_1} + \frac{1}{\bar{u}_2}$, there are multiple Robust SPNEs.*
4. *If $|\bigcap_{j=1}^{J} P_j^e| \geq 3$ then a Robust SPNE might not exists. Moreover, even if it exists it might not be unique.*

The result implies that when the intersection of efficient groups is not empty, some intermediaries have market power to charge a positive cost. When this intersection contain a single intermediary, there is a unique robust SPNE. When there are two intermediaries in this intersection, the necessary and sufficient conditions for the existence of a unique Robust SPNE are provided based on constants coming from the utility function. Moreover, when three of more intermediaries exists, multiplicity of Robust SPNE might also occur. Proposition 2 will provide sufficient conditions for the uniqueness of Robust SPNE based on constants similar to part (3).

Proof. 1. Assume $P = \{C_1, \ldots, C_k\}$. We prove that $c_n = 0$ $\forall n \geq k+1$. Indeed, assume this is not the case, that is for some $i \geq k+1$ we have that $c_i > 0$. Then, there exists group P_j^e such that $C_i \notin P_j^e$. From (2) of Proposition 3, $\exists P_1(c)$, $\forall c_n > 0$, $C_n \in P_1(c)$, and $P_1(c)$ is used by planner as a utility maximization group. Assume the maximal utility of group $P_1(c)$ and P_j^e to distribute 1

[12] $\bar{u}_{12}$ is the maximal utility of using any group of intermediaries to distribute 1 unit of resource. $\bar{u}_1$, $\bar{u}_2$ and $\bar{u}$ are the maximal utility without using C_2, C_1 and $\{C_1, C_2\}$ to distribute 1 unit of resource respectively. That is, $\bar{u}_{12} = \max u(S^T x)$, s.t $\mathbf{1}'x \leq 1$, $\mathbf{1} = (1, \ldots, 1)_{N \times 1}$. $\bar{u}_1 = \max u(S^T x)$, s.t $\mathbf{1}'x \leq 1$ and $x_2 = 0$. $\bar{u}_1$, $\bar{u}$ are defined similarly. We formalize the general case in the next section.

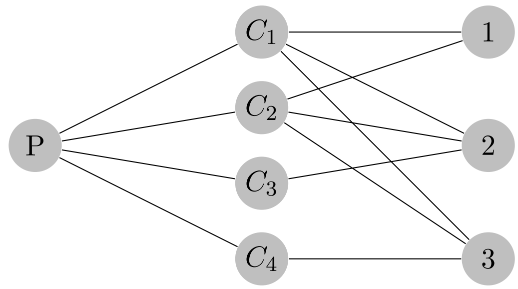

Figure 11: Network with Two Perfect Complements Intermediaries

unit of resource are u^1 and u^e respectively. $u^e \geq u^1$, $I - \sum_{C_n \in P_j^e} c_n > I - \sum_n c_n$, since $C_i \notin P_j^e$. Then $(I - \sum_{C_n \in P_j^e} c_n)u^e > (I - \sum_n c_n)u^1$, paying the cost and using group P_j^e achieve a strictly higher utility, contradiction with P maximize the utility. So $c_n = 0$, $\forall n \geq k+1$.

2. When $|\bigcap_{j=1}^J P_j^e| = 0$, the unique robust SPNE $c = 0$ is proved in Proposition 2. Assume $\bigcap_{j=1}^J P_j^e = C_1$, there is SPNE with $c = (c_1, 0, ..., 0)$, and $c_1 > 0$ satisfies $v(0, I - c_1) = v_{-1}(0, I)$. Note: 0 of $v(0, I - c_1)$ is in $\mathbb{R}^M$, 0 of $v_{-1}(0, I)$ is in $\mathbb{R}^{M-1}$. In this case, the planner could achieve utility $v_{-1}(0, I) = v(0, I | C \backslash \bigcap_{j=1}^J P_j^e)$. Moreover, if the preferences are homothetic, there exists utility function $u(Q, I) = I \cdot u(Q)$, this SPNE is the unique robust SPNE. The proof is similar to the case $\bigcap_{j=1}^J P_j^e = \emptyset$. Assume there exists intermediary $C_n \neq C_1$ charges $c_n > 0$, from (2) of conditions for SPNE, $\exists P_1$ which is a utility maximization group including all the intermediaries with positive cost. Because $\bigcap_{j=1}^J P_j^e = C_1$, $\exists P_k^e$, s.t $C_n \notin P_k^e$, the same way as Proposition to show that using group P_k^e would achieve higher utility than group P_1. Contradiction with the assumption. Thus, $c = (c_1, 0, ..., 0)$ is the unique robust SPNE.

3. $\bar{u}_{12} > \bar{u}_1$ and $\bar{u}_{12} > \bar{u}_2$. From (1), we have $c_n = 0$, $\forall n \geq 3$ in a Robust SPNE, only c_1 and c_2 may be positive. If $c_1 > 0$, $c_2 > 0$, from condition (2) of Theorem 3, $\exists$ utility maximization group $P_1(c)$ s.t $\{C_1, C_2\} \subseteq P_1(c)$. If $c_1 = 0$, $c_2 > 0$, $\exists$ utility maximization group $P_2(c)$ s.t $\{C_2\} \subseteq P_2(c)$, but $(I - c_2)\bar{u}_{12} > (I - c_2)\bar{u}_2$, thus $C_1 \subseteq P_2(c)$. So there always exists utility maximization group $P(c)$, s.t $\{C_1, C_2\} \subseteq P(c)$. Assume P_1, P_2 and P_{12} are utility maximization groups without using C_2, C_1 and $\{C_1, C_2\}$ respectively. There are two cases in a robust SPNE: 1, $P(c)$ and P_{12} are utility maximization groups. $(I - c_1 - c_2)\bar{u}_{12} = I \cdot \bar{u} > (I - c_n)\bar{u}_n$, $n = 1, 2$. When $\frac{1}{\bar{u}_{12}} + \frac{1}{\bar{u}} < \frac{1}{\bar{u}_1} + \frac{1}{\bar{u}_2}$, $c = (c_1, c_2, 0, \ldots, 0)$, with $c_1 > (1 - \frac{\bar{u}}{\bar{u}_1})I$, $c_2 > (1 - \frac{\bar{u}}{\bar{u}_2})I$, $c_1 + c_2 = (1 - \frac{\bar{u}}{\bar{u}_{12}})I > (2 - \frac{\bar{u}}{\bar{u}_1} - \frac{\bar{u}}{\bar{u}_2})I$, the planner use group $P(c)$ to transmit resource are robust SPNEs. There are multiple equilibria. 2, When $\frac{1}{\bar{u}_{12}} + \frac{1}{\bar{u}} \geq \frac{1}{\bar{u}_1} + \frac{1}{\bar{u}_2}$, $P(c)$, P_1 and P_2 are utility maximization groups. $c = (c_1, c_2, 0, \ldots, 0)$, $c_1 = \frac{\bar{u}_1\bar{u}_{12} - \bar{u}_1\bar{u}_2}{\bar{u}_{12}\bar{u}_1 + \bar{u}_{12}\bar{u}_2 - \bar{u}_1\bar{u}_2}$, $c_2 = \frac{\bar{u}_2\bar{u}_{12} - \bar{u}_1\bar{u}_2}{\bar{u}_{12}\bar{u}_1 + \bar{u}_{12}\bar{u}_2 - \bar{u}_1\bar{u}_2}$, the planner use group $P(c)$ to transmit resource is the unique robust SPNE.

When $\{C_1, C_2\}$ are perfect complements in efficient group, which means $\bar{u}_1 = \bar{u}_2 = \bar{u}$, then $\frac{1}{\bar{u}_{12}} + \frac{1}{\bar{u}} < \frac{1}{\bar{u}_1} + \frac{1}{\bar{u}_2}$. So there are multiple robust SPNEs.

4. When $|\bigcap_{j=1}^J P_j^e| \geq 3$, there might not exist Robust SPNE. If there is robust SPNE, it might not be unique like the example of perfect complements in efficient group of intermediaries in part (3). Example 17 has 3 intermediaries in the intersection of efficient groups with multiple equilibria. Moreover, the multiple robust SPNE give the planner different utility. ■

Example 13 (Multiple Robust SPNE: Perfect Complements)
Consider the network in Figure 11. Assume utility function is $u(y) = min\{y_1, y_2, y_3\}$, the matrix

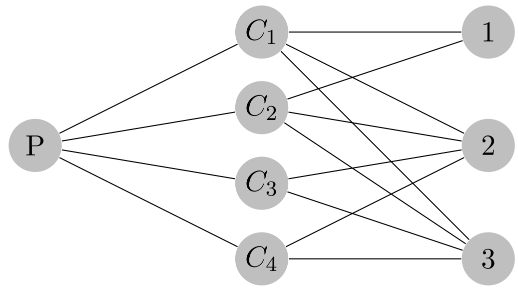

Figure 12: Network with Two Intermediaries in Intersect of Efficient Group

of sharing-rates $S = \begin{bmatrix} \frac{2}{3} & \frac{1}{6} & \frac{1}{6} \\ \frac{5}{6} & \frac{1}{12} & \frac{1}{12} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. *This example is similar to example 11, the only difference is* C_3 *in example 11 is replaced by* C_3 *and* C_4. *There are two efficient groups* $P_1^e = \{C_1, C_3, C_4\}$, $P_2^e = \{C_2, C_3, C_4\}$. $P = \bigcap_{j=1}^{J} P_j^e = \{C_3, C_4\}$. *The utility of transmitting 1 unit of resource with group* $\{C_1, C_3, C_4\}$, $\{C_1, C_3\}$, $\{C_1, C_4\}$, $\{C_1\}$ *are* $\bar{u}_{34} = \frac{1}{3}$, $\bar{u}_3 = \bar{u}_4 = \bar{u} = \frac{1}{6}$. C_3 *and* C_4 *are perfect complements intermediaries in efficient group, there are multiple robust SPNEs. In robust SPNE, the cost* $c = (0, 0, c_3, c_4)$ *s.t* $c_3 + c_4 = \frac{I}{2}$, *group* $\{C_1, C_3, C_4\}$ *are used by planner, utility of planner is* $u = \frac{I}{6}$. *The robust SPNEs are equivalent to the robust SPNE in example 11 for the planner, in which one intermediary replaces* C_3 *and* C_4 *with efficient sharing-rates.*

Example 14 (Unique Robust SPNE)
Consider the network in Figure 12. Assume utility function is $u(y) = min\{y_1, y_2, y_3\}$, *the matrix of sharing-rates* $S = \begin{bmatrix} \frac{2}{3} & \frac{1}{6} & \frac{1}{6} \\ \frac{5}{6} & \frac{1}{12} & \frac{1}{12} \\ 0 & \frac{1}{3} & \frac{2}{3} \\ 0 & \frac{2}{3} & \frac{1}{3} \end{bmatrix}$.

The network is similar to example 13 except $\{C_3, C_4\}$ *are not perfect complements in efficient group here. There are two efficient groups* $P_1^e = \{C_1, C_3, C_4\}$, $P_2^e = \{C_2, C_3, C_4\}$. $\bigcap_{j=1}^{J} P_j^e = \{C_3, C_4\}$.

The maximal utility of transmitting 1 unit of resource by group $\{C_1, C_3, C_4\}$, $\{C_1, C_3\}$, $\{C_1, C_4\}$, C_1 *are* $\bar{u}_{34} = \frac{1}{3}$, $\bar{u}_3 = \frac{4}{15}$, $\bar{u}_4 = \frac{4}{15}$, $\bar{u} = \frac{1}{6}$. *Then* $\frac{1}{\bar{u}_{34}} + \frac{1}{\bar{u}} = 9 > \frac{1}{\bar{u}_3} + \frac{1}{\bar{u}_4} = \frac{15}{2}$, *so there is unique robust SPNE. In robust SPNE, cost* $c = (0, 0, \frac{I}{6}, \frac{I}{6})$. *The utility for planner would be in* $\frac{2I}{9}$, *which is larger than* $\frac{I}{6}$ *in example 13.*

From the results of example 11, 13, 14, we find that when an intermediary in the intersect of efficient groups is separated into two perfect complements intermediaries, it brings multiple robust SPNEs, these SPNEs are welfare equivalent for the planner. When an intermediary in the intersect of efficient groups is separated into two intermediaries who are not perfect complements, it might bring competition between the intermediaries and result in an improvement for the planner.

For the case of $|\bigcap_{j=1}^{J} P_j^e| \geq 3$, the existence of SPNE becomes not clear, and there might be multiple equilibria. The following example shows multiple robust SPNE in this case.

Example 15 (Multiple Robust SPNE)
Consider the network in Figure 13. Assume utility function is $u(y) = min\{y_1, y_2, y_3, y_4\}$, *the*

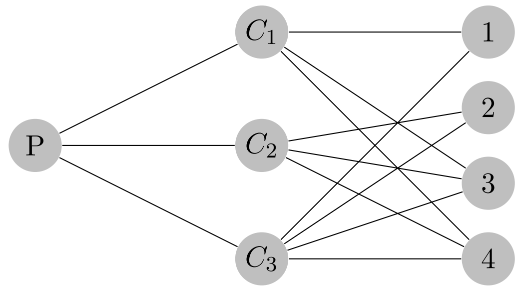

Figure 13: Network with Multiple Robust SPNE

matrix of sharing-rates $S = \begin{bmatrix} \frac{2}{3} & 0 & \frac{1}{6} & \frac{1}{6} \\ 0 & \frac{2}{3} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{6} & \frac{1}{6} \end{bmatrix}$. *There is one efficient groups* $P_1^e = \{C_1, C_2, C_3\}$. *In this example,* C_1 *and* C_2 *are perfect complements in efficient group, from (4) of Proposition 1, there are multiple robust SPNEs. In robust SPNE,* $\{C_1, C_2, C_3\}$, $\{C_1, C_2\}$, $\{C_3\}$ *are the utility maximization groups,* $\bar{u} = \frac{1}{4}$, $\bar{u}_{12} = \frac{1}{6}$, $\bar{u}_3 = \frac{1}{6}$, *the utility of planner using these groups is the same, then we have* $\frac{I-c_1-c_2-c_3}{4} = \frac{I-c_1-c_2}{6} = \frac{I-c_3}{6}$. *So any cost allocation* (c_1, c_2, c_3) *in a robust SPNE satisfies* $c_1 + c_2 = c_3 = \frac{I}{4}$, *and* $b(c) = (1, 1, 1)$, $x(c) = (\frac{I}{8}, \frac{I}{8}, \frac{I}{4})$.

As this example shows, the intersection of the efficient group includes 3 intermediaries. There are multiple robust SPNEs, but two intermediaries are perfect complements in the efficient group, and all robust SPNEs are welfare equivalent for planner.

4.3 Sufficient Conditions for Existence of SPNE

Assume the preferences are homothetic and monotonic. Denote $P = \bigcap_{j=1}^{J} P_j^e$, and $\bar{u}_Q$ the maximal utility that the planner could achieve by using group $Q \subset P$ and any other intermediaries in P^c to distribute 1 unit of resource.[13] When $Q = P$, the maximal utility is $\bar{u}_P$. We denote by $\bar{u}_{-j} = \bar{u}_{P\setminus\{j\}}$, and $\bar{u} = \bar{u}_{\emptyset}$.

The following result extends Proposition 1(3), which provides the necessary and sufficient conditions for the existence of a unique Robust SPNE in the case of $|P| = 2$. We show that these conditions can be extended for an arbitrary size of P to guarantee the uniqueness of Robust SPNE.[14]

Proposition 2 (Existence of SPNE)
Suppose $P = \bigcap_{j=1}^{J} P_j^e = \{C_1, \ldots, C_k\}$, *preferences are homothetic and monotonic and there is a vector* $(c_1, \ldots, c_k, 0, \ldots, 0)$ *such that:*

A1. $(c_1, \ldots, c_k)$ *solves the equations* $(I - \sum_{n=1}^{k} c_n)\bar{u}_P = (I - \sum_{n=1, n\neq j}^{k} c_n)\bar{u}_{-j}$ *for any* $j \in P$.

A2. $(I - \sum_{n=1}^{k} c_n)\bar{u}_P \geq (I - \sum_{n\in Q} c_n)\bar{u}_Q$, $\forall Q \subset P$.

Then, $(c_1, \ldots, c_k, 0, \ldots, 0)$ *is a SPNE. Moreover,* $(c_1, \ldots, c_k, 0, \ldots, 0)$ *is the unique Robust SPNE.*

[13]Formally, $\bar{u}_Q = \max u(S^T x)$ such that $\mathbf{1}^T x \leq 1$ and $x_n = 0$ for $C_n \in P\setminus Q$. This is a constrained optimization problem, and it can be easily solved

[14]We conjecture that these conditions are also necessary for the existence of a unique Robust SPNE (Proposition 1(3) for the case of $|P| = 2$ and all the examples in this paper support it).

Condition (A1) helps to calculate the vector of costs $(c_1, \ldots, c_k)$. Since P is the intersection of the efficient groups, then $(I - \sum_{n=1}^{k} c_n)\bar{u}_P$ is the maximal utility that the planner can achieve in the resource transmission problem after paying intermediaries in P. $(I - \sum_{n=1, n\neq j}^{k} c_n)\bar{u}_{-j}$ represent the maximal utility that the planner can achieve without using intermediary j after paying intermediaries in $P \setminus \{j\}$. Therefore, condition (A1) requires that these two utilities equal to each other. Intuitively, this means that no intermediary in P has sufficient market power, that is, after removing any agent from P the same utility can be achieved by the planner.

Condition (A2) guarantees that the maximal utility (after paying intermediaries) can be achieved by using group with P.

Note that $\bar{u}_Q$ for any $Q \subset P$ are numbers that are simple to compute from the utility function and the network. Therefore, unlike the conditions from the previous Theorems (such as Theorem 2), this result provides sufficient conditions for the existence of SPNE and Robust SPNE that can be computed easily as a system of linear equations and linear inequalities. The explicit requirements on the utility function imposed by condition A1 and A2 are provided in Corollary 1, which solves the above system of equations for a general number of intermediaries in P.

Proof. To start, first note that if any intermediary $C_n \notin P$ increase cost from 0 to positive, then it would not be used, since we can find an efficient group P_j^e, s.t $C_n \notin P_j^e$.

Only the intermediaries $C_1, \ldots, C_k$ are used by planner, thus none of them have the incentive to decrease their costs. For C_n, $1 \leq n \leq k$, the utility of efficient group is the same as a group without C_n in condition (1), if it increases cost, C_n would not be used and lose all the payment from planner.

Thus, no intermediary has incentive to deviate, $(c_1, \ldots, c_k, 0, \ldots, 0)$ is SPNE.

To show the uniqueness of Robust SPNE, in condition (A1) of Proposition 1, Assume cost $c^* = (c_1^*, \ldots, c_k^*)$ solves the equations $(I - \sum_{n=1}^{k} c_n^*)\bar{u}_P = (I - \sum_{n=1, n\neq j}^{k} c_n^*)\bar{u}_{-j}$. $(c_1^*, \ldots, c_k^*, 0, \ldots, 0)$ is SPNE. From Proposition 1 (1), for any other robust SPNE, $c = (c_1, \ldots, c_N)$, there is $c_n = 0$, $\forall n \geq k+1$. Then prove the utility maximization group $P_1(c)$, $\forall c_n > 0$, $C_n \in P_1(c)$ is an efficient group. If $P_1(c)$ is not an efficient group, assume planner's maximal utility of transmit 1 unit of resource using group $P_1(c)$ is u_1, compared to utility $\bar{u}_P$ of transmitting with efficient group, $u_1 < \bar{u}_P$. Then there is $(I - \sum_n c_n)u_1 < (I - \sum_n c_n)\bar{u}_P$, contradiction with $P_1(c)$ is a utility maximization group. Then $\exists P_j^e$, $P_1(c) = P_j^e$. In robust SPNE, since $P_1(c) = P_j^e$, we have $(I - \sum_{n=1}^{k} c_n)\bar{u}_P \geq (I - \sum_{n=1, n\neq j}^{k} c_n)\bar{u}_{-j}$, which is equivalent with $c_j \leq (I - \sum_{n=1}^{k} c_n)(\frac{\bar{u}_P}{\bar{u}_{-j}} - 1)$.

If $\sum_{n=1}^{k} c_n > \sum_{n=1}^{k} c_n^*$, $\exists c_j > c_j^*$, there is $I - \sum_{n=1}^{k} c_n^* > I - \sum_{n=1}^{k} c_n$, then $c_j > c_j^* = (I - \sum_{n=1}^{k} c_n^*)(\frac{\bar{u}_P}{\bar{u}_{-j}} - 1) > (I - \sum_{n=1}^{k} c_n)(\frac{\bar{u}_P}{\bar{u}_{-j}} - 1)$, contradiction.

If $\sum_{n=1}^{k} c_n = \sum_{n=1}^{k} c_n^*$, then $c_n = c_n^*$. Otherwise, $\exists c_j > c_j^*$, same with the case of $\sum_{n=1}^{k} c_n > \sum_{n=1}^{k} c_n^*$, contradiction.

If $\sum_{n=1}^{k} c_n < \sum_{n=1}^{k} c_n^*$. In robust SPNE, $(I - \sum_{n=1}^{k} c_n)\bar{u}_P \geq (I - \sum_{n=1, n\neq j}^{k} c_n)\bar{u}_{-j}$, it is equivalent with $\frac{c_j}{I - \sum_{n=1}^{k} c_n} \leq \frac{\bar{u}_P}{\bar{u}_{-j}} - 1$, $\forall j$. The equality does not hold for all j, otherwise $c = c^*$. $\exists j$, $(I - \sum_{n=1}^{k} c_n)\bar{u}_P > (I - \sum_{n=1, n\neq j}^{k} c_n)\bar{u}_{-j}$, which means any group P_2 with $P_2 \cap P = P \backslash \{j\}$ is not a utility maximization group. From (1) of Proposition 3, we have $\bigcap P_l(c) = \emptyset$, so $\exists P_i(c)$, s.t $C_j \notin P_i(c)$, and $P_i(c) \cap P \neq P \backslash \{j\}$, assume $P_i(c) \cap P = Q_i(c)$. $P_i(c)$ is utility maximization group, then $(I - \sum_{n=1}^{k} c_n)\bar{u}_P = (I - \sum_{C_n \in Q_i(c)} c_n)\bar{u}_{Q_i(c)}$, equivalent with $\frac{\sum_{C_n \in Q_i(c)} c_n}{I - \sum_{n=1}^{k} c_n} = \frac{\bar{u}_P}{\bar{u}_{Q_i(c)}} - 1$. From condition (A2), we have $(I - \sum_{n=1}^{k} c_n^*)\bar{u}_P > (I - \sum_{C_n \in Q_i(c)} c_n^*)\bar{u}_{Q_i(c)}$, equivalent with $\frac{\sum_{C_n \in Q_i(c)} c_n^*}{I - \sum_{n=1}^{k} c_n^*} < \frac{\bar{u}_P}{\bar{u}_{Q_i(c)}} - 1$. Then there is $\frac{\sum_{C_n \in Q_i(c)} c_n^*}{I - \sum_{n=1}^{k} c_n^*} \leq \frac{\bar{u}_P}{\bar{u}_{Q_i(c)}} - 1 = \frac{\sum_{C_n \in Q_i(c)} c_n}{I - \sum_{n=1}^{k} c_n}$. From condition (A1), $\frac{c_j^*}{I - \sum_{n=1}^{k} c_n^*} = \frac{\bar{u}_P}{\bar{u}_{-j}} - 1$, $\forall j$. Then $\frac{c_j^*}{I - \sum_{n=1}^{k} c_n^*} = \frac{\bar{u}_P}{\bar{u}_{-j}} - 1 > \frac{\bar{u}_P}{\bar{u}_{Q_i(c)}} - 1 \geq \frac{c_j}{I - \sum_{n=1}^{k} c_n}$, $\forall j$. Sum over

$C_j \in Q_i(c)$, contradiction in the equality.

$(c_1, \ldots, c_k, 0, \ldots, 0)$ satisfies condition (A1) is the unique robust SPNE. ■

Corollary 1 (Explicit conditions for uniqueness of Robust SPNE)
Let $P = \{C_1, \ldots, C_k\}$. Assume that preferences are homothetic and monotonic. If the following conditions are satisfied, then a unique robust SPNE exists:

$k = 3$

a. $\frac{1}{\bar{u}_{123}} + \frac{1}{\bar{u}_3} \geq \frac{1}{\bar{u}_{23}} + \frac{1}{\bar{u}_{13}}$, *along with symmetric conditions for intermediaries 1 and 2.*

b. $\frac{2}{\bar{u}_{123}} + \frac{1}{\bar{u}} \geq \frac{1}{\bar{u}_{23}} + \frac{1}{\bar{u}_{13}} + \frac{1}{\bar{u}_{12}}$

$k = 4$

a1. *The other condition* $\frac{2}{\bar{u}_{1234}} + \frac{1}{\bar{u}_4} \geq \frac{1}{\bar{u}_{234}} + \frac{1}{\bar{u}_{134}} + \frac{1}{\bar{u}_{124}}$ *along with the symmetric conditions for 1, 2 and 3.*

a2. $\frac{1}{\bar{u}_{1234}} + \frac{1}{\bar{u}_{34}} \geq \frac{1}{\bar{u}_{234}} + \frac{1}{\bar{u}_{134}}$, *along with the symmetric conditions for any group of two intermediaries in P.*

b. $\frac{3}{\bar{u}_{1234}} + \frac{1}{\bar{u}} \geq \frac{1}{\bar{u}_{123}} + \frac{1}{\bar{u}_{234}} + \frac{1}{\bar{u}_{124}} + \frac{1}{\bar{u}_{134}}$.

$k = K$

a. *For any $i \geq 0$:* $\frac{K-i-1}{\bar{u}_{1\ldots K}} + \frac{1}{\bar{u}_{(K-i+1)\ldots K}} \geq \frac{1}{\bar{u}_{-1}} + \cdots + \frac{1}{\bar{u}_{-(K-i)}}$, *along with the symmetric conditions for any group of i intermediaries in P.*

b. $\frac{K-1}{\bar{u}_{1\ldots K}} + \frac{1}{\bar{u}} \geq \frac{1}{\bar{u}_{-1}} + \cdots + \frac{1}{\bar{u}_{-K}}$.

This corollary is important because it provides sufficient conditions, that are directly computable from the utility function, which guarantee the uniqueness of the Robust SPNE.

The cases of $K = 3$ and $K = 4$ are provided for illustrative purposes only, but they are clearly a particular case of general case $k = K$. In this case, conditions (a) can basically be implied from condition (A2) above, when the deviating group is $Q = \{C_{K-i+1} \ldots C_K\}$. Also, note that conditions (b) is very similar to condition (a), for the case of $Q = \emptyset$.

Proof.

Notice condition (A1) determines a unique set of costs $c_1, \ldots c_k$, since that is a system of k linearly independent equations with k unknowns. We show that if conditions (a) and (b) are satisfied, then the costs $c_1, \ldots c_k$ satisfy condition (A2). We prove this claim based on induction in the size of P. The base on induction, $k = 2$, is proved in Proposition 1(3). Suppose that the statement hold for $k \leq K - 1$. We will show it for $k = K$.

Case 1. Q such that $\emptyset \neq Q \subset P$. In this case, the inequality (A2) is $(I - \sum_{n=1}^{k} c_n)\bar{u}_P \geq (I - \sum_{n \in Q} c_n)\bar{u}_Q$.

Without loss of generality, assume $Q = \{C_{K-i}, \ldots, C_K\}$. Then, (a) implies $\frac{K-i-2}{\bar{u}_{1\ldots K}} + \frac{1}{\bar{u}_{(K-i)\ldots K}} \geq \frac{1}{\bar{u}_{-1}} + \frac{1}{\bar{u}_{-2}} + \cdots + \frac{1}{\bar{u}_{-(K-i-1)}}$. And inequality in (A2) implies $(I - \sum_{n=1}^{K} c_n)\bar{u}_P \geq (I - \sum_{n=K-i}^{K} c_n)\bar{u}_{(K-i)\ldots K}$. Consider the case $k = K - i$, $Q = \{C_{K-i}\}$. From induction, the inequality from (A2) $(I - \sum_{n=1}^{K-i} c_n)\bar{u}_P \geq (I - c_{K-i})\bar{u}_{K-i}$ is equivalent with inequality of (a) $\frac{K-i-2}{\bar{u}_{1\ldots(K-i)}} + \frac{1}{\bar{u}_{K-i}} \geq \frac{1}{\bar{u}_{-1}} + \frac{1}{\bar{u}_{-2}} + \cdots + \frac{1}{\bar{u}_{-(K-i-1)}}$. Take the intermediaries $Q = \{C_{K-i}, \ldots, C_K\}$ in case $k = K$ as a group, the inequalities of (A2) and (a) corresponds to inequalities in the case of $Q = \{C_{K-i}\}$ and $k = K-i$. Since $(I - \sum_{n=1}^{K-i} c_n)\bar{u}_P \geq (I - c_{K-i})\bar{u}_{K-i}$ is equivalent with $\frac{K-i-2}{\bar{u}_{1\ldots(K-i)}} + \frac{1}{\bar{u}_{K-i}} \geq \frac{1}{\bar{u}_{-1}} + \frac{1}{\bar{u}_{-2}} + \cdots + \frac{1}{\bar{u}_{-(K-i-1)}}$. Replace $Q = \{C_{K-i}\}$ with $Q = \{C_{K-i}, \ldots, C_K\}$, there is $(I - \sum_{n=1}^{K} c_n)\bar{u}_P \geq (I - \sum_{n=K-i}^{K} c_n)\bar{u}_{(K-i)\ldots K}$ (from (A2)) equivalent with $\frac{K-i-2}{\bar{u}_{1\ldots K}} + \frac{1}{\bar{u}_{(K-i)\ldots K}} \geq \frac{1}{\bar{u}_{-1}} + \frac{1}{\bar{u}_{-2}} + \cdots + \frac{1}{\bar{u}_{-(K-i-1)}}$ (from (a)).

Case 2. $Q = \emptyset$, which implies that condition (A2) is $(I - \sum_{n=1}^{k} c_n)\bar{u}_P \geq \bar{u}I$.

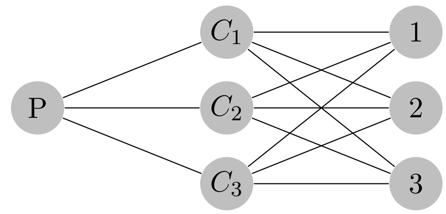

Figure 14: Network with Cobb-Douglas Utility Function

From (A1), $(I-\sum_{n=1}^{k} c_n)\bar{u}_P = (I-\sum_{n=1,n\neq j}^{k} c_n)\bar{u}_{-j}$. This is equivalent to $\frac{c_j}{I-\sum_{n=1}^{k} c_n} = \frac{\bar{u}_P}{\bar{u}_{-j}} - 1$. By summing over j, we have that $\sum_{n=1}^{k} c_n = \frac{(\sum_{n=1}^{k}\frac{\bar{u}_P}{\bar{u}_{-n}}-K)I}{1-K+\sum_{n=1}^{k}\frac{\bar{u}_P}{\bar{u}_{-n}}}$. Therefore, (A2) holds if and only if $(I-\sum_{n=1}^{K} c_n)\bar{u}_P \geq \bar{u}I$, if and only if, $\frac{\bar{u}_P}{1-K+\sum_{n=1}^{k}\frac{\bar{u}_P}{\bar{u}_{-n}}} \geq \bar{u}$. Since $k=K$, this is equivalent to $\frac{K-1}{\bar{u}_P} + \frac{1}{\bar{u}} \geq \sum_{n=1}^{K}\frac{1}{\bar{u}_{-n}}$. Thus, $\bar{u}_P = \bar{u}_{1\ldots K}$, and hence $\frac{K-1}{\bar{u}_{1\ldots K}} + \frac{1}{\bar{u}} \geq \frac{1}{\bar{u}_{-1}} + \cdots + \frac{1}{\bar{u}_{-K}}$. ■

Example 16 (Cobb-Douglas Utility Function)
The proposition above gives sufficient conditions for the existence of SPNE, the network and utility function in this example satisfy these conditions. There are 3 intermediaries in a symmetric network. Consider the network in Figure 14. Assume planner has Cobb-Douglas utility function $u(y) = y_1^{\frac{1}{3}} y_2^{\frac{1}{3}} y_3^{\frac{1}{3}}$, and the matrix of sharing-rates is $S = \begin{bmatrix} 1-2\alpha & \alpha & \alpha \\ \alpha & 1-2\alpha & \alpha \\ \alpha & \alpha & 1-2\alpha \end{bmatrix}$. Assume $0 < \alpha < \frac{1}{3}$, the maximal utility by using all intermediaries is $\bar{u}_{123} = \frac{1}{3}I$, the maximal utility of using two intermediaries is $\bar{u}_{12} = \bar{u}_{23} = \bar{u}_{13} = (\frac{1-\alpha}{2})^{\frac{2}{3}}\alpha^{\frac{1}{3}}I$, utility of using one intermediary is $\bar{u}_1 = \bar{u}_2 = \bar{u}_3 = (1-2\alpha)^{\frac{1}{3}}\alpha^{\frac{2}{3}}I$. Substitute the utility into the condition for the existence of SPNE discussed above: $\frac{1}{\bar{u}_{123}} + \frac{1}{\bar{u}_3} \geq \frac{1}{\bar{u}_{23}} + \frac{1}{\bar{u}_{13}}$. This condition becomes $3 + \alpha^{-\frac{2}{3}}(1-2\alpha)^{-\frac{1}{3}} \geq 2^{\frac{5}{3}}\alpha^{-\frac{1}{3}}(1-\alpha)^{-\frac{2}{3}}$, which is satisfied for all $\alpha \in (0, \frac{1}{3})$.

Example 17 (Multiple Robust SPNE with Different Utility)
Consider the case with 3 intermediaries in the intersection of efficient groups, assume $P = \{C_1, C_2, C_3\}$, then $\bar{u}_{123}$ and $\bar{u}_Q$ is defined in the beginning of section 4.3. In robust SPNE, only $\{C_1, C_2, C_3\}$ might charge positive cost. There are three cases of robust SPNE, depending on the value of $\bar{u}_{123}$, $\bar{u}_1$, $\bar{u}_2$, $\bar{u}_3$, $\bar{u}_{12}$, $\bar{u}_{23}$, $\bar{u}_{13}$. The maximal utility groups could be in the following cases:

1. $\{C_1, C_2, C_3\}$, $\{C_1, C_2\}$, $\{C_1, C_3\}$, $\{C_2, C_3\}$.
2. $\{C_1, C_2, C_3\}$, $\{C_1\}$, $\{C_2, C_3\}$.
3. $\{C_1, C_2, C_3\}$, $\{C_1\}$, $\{C_2\}$.

In case 1, we could solve the cost allocation, and there is unique robust SPNE from Proposition 2. In case 2, there are multiple robust SPNE, but c_1 and c_2+c_3 are fixed in all equilibria, which is similar to the case with perfect complements in efficient group, and result in the same utility for planner. In case 3, there is $(I-c_1-c_2-c_3)\bar{u}_{123} = (I-c_1)\bar{u}_1 = (I-c_2)\bar{u}_2$, we get $c = (c_1, c_2(c_1), c_3(c_1))$, and $c_2'(c_1) > 0$, $c_3'(c_1) < 0$. The robust SPNE satisfies the utility higher than using group $\{C_1, C_2\}$, $\{C_1, C_3\}$, $\{C_2, C_3\}$, $\{C_3\}$, which means $(I-c_1-c_2-c_3)\bar{u}_{123} \geq (I-c_1-c_3)\bar{u}_{13}$, $(I-c_1-c_2-c_3)\bar{u}_{123} \geq$

$(I-c_2-c_3)\bar{u}_{23}$, $(I-c_1-c_2-c_3)\bar{u}_{123} \geq (I-c_1-c_2)\bar{u}_{12}$, $(I-c_1-c_2-c_3)\bar{u}_{123} \geq (I-c_3)\bar{u}_3$. These inequalities are the conditions of c_1 to be equilibrium. The robust SPNE in this case exists if there is c_1 satisfies all these inequalities. The results of calculation show the conditions for existence of robust SPNE are $\frac{2}{\bar{u}_{12}}+\frac{1}{\bar{u}_3} \geq \frac{1}{\bar{u}_1}+\frac{1}{\bar{u}_2}+\frac{1}{\bar{u}_{123}}$, $\frac{1}{\bar{u}_{12}}+\frac{1}{\bar{u}_{13}} \geq \frac{1}{\bar{u}_1}+\frac{1}{\bar{u}_{123}}$, $\frac{1}{\bar{u}_{23}}+\frac{1}{\bar{u}_{12}} \geq \frac{1}{\bar{u}_2}+\frac{1}{\bar{u}_{123}}$. So when $\bar{u}_1=\bar{u}_2=2$, $\bar{u}_3=1$, $\bar{u}_{12}=3$, $\bar{u}_{13}=\bar{u}_{23}=2.5$, $\bar{u}_{123}=5$, there are different c_1 in equilibria, and the utility of planner equals $(I-c_1)\bar{u}_1$, thus the utility varies in the multiple robust equilibria.

5 Comparative Statics Analysis

In this section, we study the effect of network changes in the equilibria. The following corollary provide conditions under which an efficient-SPNE and unique efficient robust SPNE will occur by adding some intermediaries to the game.

Corollary 2 (Addition of intermediaries that guarantee efficient-SPNE)
Suppose that the planner has strongly monotonic and homothetic preferences.

1. *In any network such that there are at least two intermediaries of every type, there exists an efficient SPNE and a unique Efficient Robust SPNE.*

2. *If we replicate any problem,[15] then an efficient SPNE will exists and a unique Efficient Robust SPNE will exist.*

3. *If an intermediary C_{N+1} is added such that his sharing rate can be expressed at the linear combination of the shares in the intersection of efficient groups P[16], then an efficient SPNE and a unique Efficient Robust SPNE will occur.*

This result implies that by replicating the existing intermediaries or by adding an intermediary which efficiently transmit the resources as other efficient intermediaries, will result in efficient SPNE and unique efficient robust SPNE. This generation of efficiency is independent on whether the initial network had a very inefficient equilibrium. The reason behind this corollary is similar to Theorem 1: when every intermediary can be substituted by other group of intermediaries that are equally efficient, there is equilibrium with perfect competition among the intermediaries.

Proof.

1. Assume there are N types of intermediaries $\{C_1, C_2, \ldots, C_N\}$, there are r_n intermediaries $\{C_{n(1)}, C_{n(2)}, \ldots, C_{n(r_n)}\}$ for type n. The intermediaries of type n have the same sharing-rates $s_n = (s_{n1}, \ldots, s_{nM})$, so transmitting resource of allocation $(x_{n(1)}, \ldots, x_{n(r_n)})$ with type n intermediaries is equivalent to transmitting $\sum_{j=1}^{r_n} x_{n(j)}$ with any one type n intermediary, since in either case, the resource allocated to the agents is $(\sum_{j=1}^{r_n} x_{n(j)})s_n$. For any efficient group P_1^e, assume $x = (x_{n(j)})_{1\leq n\leq N, 1\leq j\leq r_n}$ is an allocation that maximizes the planner's utility with group P_1^e. Let x' be another allocation satisfy $x'_{n(1)} = \sum_{j=1}^{r_n} x_{n(j)}$, and $x'_{n(j)} = 0$ for $j \geq 2$, then $y = S^T x = S^T x' = y'$, x' also achieves the maximal utility for planner. Since $x'_{n(j)} = 0$ for $j \geq 2$, only a subset of intermediaries $\{C_{1(1)}, C_{2(1)}, \ldots, C_{N(1)}\}$ is used, assume P_1 is the set of intermediaries with $C_{n(1)} \in P_1$ if and only if $x'_{n(1)} > 0$. P_1 is an efficient group because x' results in utility maximization allocation. Construct x'' to be efficient resource allocation similar to x'. Let $x''_{n(j)} = 0$ for $j \neq 2$, $x''_{n(2)} = x'_{n(1)}$, then $y'' = S^T x'' = S^T x = y$. Similarly define P_2, $P_2 \subseteq \{C_{1(2)}, \ldots, C_{N(2)}\}$ and

[15] That is, every intermediary is duplicated, including his connections and sharing rates.

[16] The linear combination means the sharing-rates of C_{N+1} is a linear combination of the sharing-rates vectors of intermediaries in P, the weight is determined by an allocation x maximizing planner's utility.

$C_{n(2)} \in P_2$ if and only if $x''_{n(2)} > 0$. P_2 is also an efficient group. $P_1 \cap P_2 = \emptyset$. So the intersection of efficient groups $\bigcap_{k=1}^{K} P_k(c) = \emptyset$. From Theorem 1 and Theorem 2, there is efficient-SPNE and $c = 0$ is the unique efficient robust SPNE.

2. Assume the intermediaries are $C = \{C_1, C_2, \ldots, C_N\}$, and the replicated intermediaries are $C_r = \{C_{r1}, C_{r2}, \ldots, C_{rN}\}$, C_n and C_{rn} are intermediaries of the same type, so there are at least two intermediaries of the same type. From part 1, an efficient SPNE will exists and a unique Efficient Robust SPNE will exist.

3. Assume $P = \bigcap_{j=1}^{J} P_j^e = \{C_1, \ldots, C_k\}$ and $x = (x_1, \ldots, x_N)$ is the resource allocation, which maximizes planner's utility, using efficient group P_1^e. The sharing-rates of intermediary C_n is $s_n = (s_{n1}, \ldots, s_{nM})$, the resource allocated via intermediaries $\{C_1, \ldots, C_k\}$ is $\sum_{n=1}^{k} x_n s_n = (y_1^1, \ldots, y_M^1) = y^1$. Assume the sharing-rates of intermediary C_{N+1} is $s_{N+1} = (s_{(N+1)1}, \ldots, s_{(N+1)M})$, and let $s_{(N+1)i} = \frac{y_i^1}{\sum_{m=1}^{M} y_m^1}$, or equivalently $s_{N+1} = \frac{\sum_{n=1}^{k} x_n s_n}{\sum_{n=1}^{k} x_n}$, a linear combination of sharing-rates vectors of intermediaries in P. The matrix of sharing-rates after adding C_{N+1} is S'. Construct a new resource allocation x', s.t $x'_n = 0$ for $n \leq k$, $x'_n = x_n$ for $k+1 \leq n \leq N$, and $x'_{N+1} = \sum_{m=1}^{M} y_m^1$. Then $S'^T x' = y' = y = S^T x$, which means the planner is able to achieve the prior efficient allocation with the group $P_1 = C_{N+1} \cup (P_1^e \backslash P)$. Consider an allocation $x^1 = (x_1^1, \ldots, x_{N+1}^1)$ via intermediaries $\{C_1, \ldots, C_{N+1}\}$, we can find another allocation x^2 with $x_n^2 = (\frac{x_n}{\sum_{i=1}^{k} x_i}) \cdot x_{N+1}^1 + x_n^1$ for $n \leq k$, $x_n^2 = x_n^2$ for $k+1 \leq n \leq N$, and $x_{N+1}^2 = 0$. x^1 and x^2 result in the same resource allocation, and x^2 only use intermediaries $\{C_1, \ldots, C_N\}$, so the possible resource allocations to agents are the same as before adding C_{N+1}, and maximal utility of planner will not change. P_j^e and P_j^e are efficient group and $P_1 \cap P = \emptyset$. From Theorem 1 and Theorem 2, there is efficient-SPNE and $c = 0$ is the unique efficient robust SPNE. ■

In the following corollary, we consider the change in the planner's utility of SPNE by adding one intermediary.

Corollary 3 (Comparative statics on the addition of an intermediary)
Suppose that preferences are homothetic and monotonic. Consider an arbitrary problem with N intermediaries and add the intermediary C_{N+1}. Assume $\{P_{0j}^e\}_j$ and $\{P_{1j}^e\}_j$ are the efficient groups before and after adding C_{N+1}, respectively. Let P_0 and P_1 be the intersection of the efficient groups, $P_0 = \bigcap_j P_{0j}^e$ and $P_1 = \bigcap_j P_{1j}^e$, respectively.

1. *If C_{N+1} does not belong to any new efficient group P_{1j}^e and condition (A1), (A2) satisfy, then multiplicity of equilibria may occur in the extended problem. Every robust SPNE will not be worse-off than the previous robust SPNE.*

2. *If $C_{N+1} \in P_{1j}^e$ but $C_{N+1} \notin P_1$ and condition (A1), (A2) satisfy, then multiplicity of equilibria may occur in the extended problem. Every robust SPNE will not be worse-off than the previous robust SPNE.*

3. *If $C_{N+1} \in P_{1j}^e$, $C_{N+1} \notin P_1$ and the preferences are strictly convex, then the robust SPNE will not change.*

4. *If $P_1 = \{C_{N+1}\}$ and $P_0 \neq \emptyset$, then the planner is strictly better-off.*

5. *If $C_{N+1} \in P_1$, then the planner is not necessarily better-off due to multiplicity of equilibria.*

This result shows that under the condition that there is unique robust SPNE before adding a new intermediary, the planner will not be worse off. But when there are multiple robust SPNE, the planner may be worse off or better off based on the selection of equilibrium. When the new intermediary is the unique intermediary in the intersection of the efficient groups, this intermediary has market power to charge positive cost and there is unique SPNE.

Proof.

1. If C_{N+1} does not belong to any new efficient group P^e_{1j}, the efficient groups are the same, $P_0 = P_1 = \{C_1, \dots, C_k\}$, then $c'_j = 0$ for $j \geq k$ in a robust SPNE. The maximal utility $\bar{u}$ of planner will not change. Condition (A1), (A2) satisfy, there is unique robust SPNE, the cost allocation satisfy (A1) $(c_1, \dots, c_k)$ solves the equations $(I - \sum_{n=1}^k c_n)\bar{u} = (I - \sum_{n=1, n\neq j}^k c_n)\bar{u}_{-j}$, thus $c_j = (I - \sum_{n=1}^k c_n)(\frac{\bar{u}}{\bar{u}_{-j}} - 1)$. For $j \leq k$, because the efficient group would be used in SPNE and its utility will be no less than group without intermediary C_j, $(I - \sum_{n=1}^k c'_n)\bar{u} \geq (I - \sum_{n=1, n\neq j}^k c'_n)\bar{u}'_{-j}$, thus $c'_j \leq (I - \sum_{n=1}^k c'_n)(\frac{\bar{u}}{\bar{u}'_{-j}} - 1)$. By definition, we will have $\bar{u}'_{-j} \geq \bar{u}_{-j}$. If $\sum_{n=1}^k c'_n > \sum_{n=1}^k c_n$, $\exists c'_j > c_j$, there is $I - \sum_{n=1}^k c_n > I - \sum_{n=1}^k c'_n$, then $c'_j > c_j = (I - \sum_{n=1}^k c_n)(\frac{\bar{u}}{\bar{u}_{-j}} - 1) > (I - \sum_{n=1}^k c'_n)(\frac{\bar{u}}{\bar{u}'_{-j}} - 1)$, contradiction. So total cost $\sum_{n=1}^k c'_n \leq \sum_{n=1}^k c_n$, the robust SPNE will not be worse off than previous robust SPNE.

2. If $C_{N+1} \in P^e_{1j}$, $C_{N+1} \notin P_1$. Assume the maximal utility of allocation given cost $c = 0$ is $\bar{u}$. $C_{N+1} \notin P_1$, then the maximal utility after adding C_{N+1} is still $\bar{u}$, otherwise C_{N+1} will be in every efficient group P^e_{1j}, thus the efficient group P^e_{0j} is still efficient group. After adding $C_{N+1} \notin P_1$, $P_1 \subseteq P_0$, then $c'_j = 0$ for $j \geq k$ in a robust SPNE. Before adding C_{N+1}, there is unique robust SPNE, the cost allocation satisfy (A1), the proof of part 1 follows.

3. When the preferences are strictly convex, there is unique allocation y maximize the utility of planner with $c = 0$. The intermediaries in P_1 will not be linear represented by other intermediaries. $C_{N+1} \in P^e_{1j}$ and $C_{N+1} \notin P_1$, so there is x^1 and x^2, such that $y = S^T x^1 = S^T x^2$, x^1 use C_{N+1} and x^2 not use C_{N+1}. Then the sharing-rates s_{N+1} of C_{N+1} is linear combination of sharing-rates of some intermediaries in $C \backslash P$. By definition of $\bar{u}_Q$, it will not change after adding C_{N+1}, $\bar{u}_Q = \bar{u}'_Q$. The cost allocation in a robust SPNE only depends on the value of $\bar{u}'_Q$, so the robust SPNE will not change.

4. If $P_1 = \{C_{N+1}\}$, assume the maximal utility for planner transmitting I units of resource without paying cost is $\bar{u}$, and maximal utility without using C_{N+1} is $\bar{u}_0$. From part 2 of Proposition 1, $c_n = 0$ for $n \leq N$, and c_{N+1} satisfies $(I - c_{N+1}) \cdot \bar{u} = \bar{u}_0$, planner's utility is $\bar{u}_0$. If $P_0 \neq \emptyset$, the utility of planner would be lower than $\bar{u}_0$, then the planner is strictly better off.

5. From example 17, there are multiple robust SPNE, which result in different utility for planner. If $C_{N+1} \in P_1$, the robust SPNE might be very different with the robust SPNE before adding C_{N+1}. Due to the multiplicity of equilibria, the planner is not necessarily better-off. ■

6 Flexible Sharing-Rates

In this section we extend the mechanism discussed above to the case where the planner might select not only the intermediaries and allocation but also determine the sharing-rates.

Formally, in a *Flexible Sharing-Rates* model, a network of connections between the intermediaries and agents, and the vector of waste-rates $(d_1, \dots, d_N)$ of the intermediaries is given. The intermediaries choose the cost c to charge the planner for using their links. The planner chooses the intermediaries to be used, the matrix of sharing-rates $S(c) = (s_{11}, \dots, s_{NM})_{N\times M}$ such that $\sum_{m=1}^N s_{nm} = d_n$ for any $n \in 1, \dots, M$, and the allocation $x(c)$ to the agents. This assumption is natural when the planner has the flexibility to determine the sharing-rates, subject to the constraint of paying the waste-rates of every intermediary. A similar model, without waste-rates, has been studied in spanning trees, see Moulin and Velez[46].

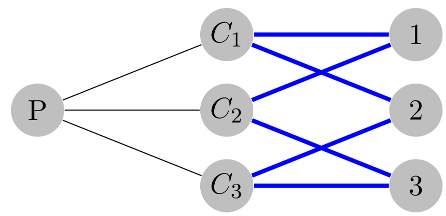

Figure 15: Network with Flexible Sharing-Rates

Example 18 (Multiple Equilibria)
Consider the network in Figure 15. The graph $G = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$. *Assume the planner has Cobb-Douglas utility function* $u(y) = y_1^{\frac{1}{3}} y_2^{\frac{1}{3}} y_3^{\frac{1}{3}}$. *The utility of using one intermediary is 0, and the utility of using 2 intermediaries to transmit* L *units of resource is* $\frac{L}{3}$. *The planner would always choose the group of two intermediaries who charge the lowest total cost. The intermediary who is not selected by planner has the incentive to decrease its cost in order to get selected. This behavior of Bertrand-undercutting always happens except in the case where lowering cost won't bring the total cost below the total level of resources. Thus, there are SPNEs with* $c = (0, 0, 0)$, *and* $c = (I, I, I)$. *It is easy to verify that both of them can be cost allocations in SPNE.*

Definition 7 (Waste Rates of Link to Agents)
Suppose that the waste rates of intermediaries are $d = (d_1, \ldots, d_N)$. *The rate of resource transmitted to agent* $m \in \{1, \ldots, M\}$ *with intermediary* C_n *is* $d_n g_{nm}$. *Let* $d^1(m) \geq \cdots \geq d^N(m)$ *be the rates of resource transmission from intermediaries to agent* m, *sorted from larger to smaller.*

Theorem 4 (Necessary and sufficient conditions for the existence of an Efficient SPNE)
Suppose that the utility function of the planner has non-zero corners.[17]

1. *If* $d_1 = d_2 = \cdots = d_N$ *and there are at least 2 links to each agent,* $\sum_{n=1}^{N} g_{nm} \geq 2$, $\forall m$, *then there is efficient SPNE with* $c = 0$. *Moreover, the SPNEs would be either (a) the intermediaries chosen by planner charge cost 0 (the efficient equilibrium); or (b) the intermediaries chosen charge total cost no less than* I, *thus no resource is transmitted to the agents.*

2. *If* $d_1 = d_2 = \cdots = d_N$ *and there is one agent served by a unique intermediary, then this intermediary charges* I *at every SPNE and no resource is transmitted to the agents.*

3. *If the preferences are strongly monotone, there is an efficient SPNE if and only if* $d^1(m) = d^2(m)$, $\forall m$.

This result implies that when the capacity of every link is equal and there are at least two links to each agent — that is, every link could be substituted by another link, there is an equilibrium with perfect competition among intermediaries. The cost asked by intermediaries would be either the efficient cost ($c = 0$) or the fully inefficient equilibrium where intermediaries charge all the resource. The assumption of zero corners is very relevant because when there is a monopoly in the transmission of the resource to an agent —that is an intermediary is the unique connection to an agent, this monopolist intermediary has the ability to charge all the resources of the planner.

[17] We say that the utility function has non-zero corners if for any vector $y \in \mathbb{R}_+^M$ such that $\exists y_m = 0$ then $u(y) = 0$; and if $y > 0$, then $u(y) > 0$. $u(y) = min\{y_1, \ldots, y_M\}$ and Cobb-Douglas utility function $u(y) = \prod_{m=1}^{M} y_m^{\alpha_m}$ are examples that satisfy this condition.

Proof.

1. First we show the existence of an efficient SPNE with $c = 0$. Since $d_1 = d_2 = \cdots = d_N$, any group of intermediaries who collectively can link to all the agents is an efficient group. Assume that the efficient groups are $P_1^e, \dots, P_J^e$. Without loss of generality, intermediary $C_1 \subseteq P_1^e$ with the links to agents $1, \dots, k$, which is $g_{11} = \cdots = g_{1k} = 1$. Since $\sum_{n=1}^N g_{nm} \geq 2$, there exists groups of intermediaries $Q_1 = \{C^1, \dots, C^k\}$ (the intermediaries may repeat) and C^i has link to agent i, $\forall 1 \leq i \leq k$. Thus $P_2 = P_1^e \cup Q_1 \backslash C_1$ is also an efficient group. Thus there is $\bigcap_{j=1}^J P_j^e = \emptyset$. Given $c_{-n} = 0$, when C_n choose $c_n > 0$, the planner will use an efficient without C_n, there is no incentive for C_n to deviate $c_n = 0$. Thus, $c = 0$ is a SPNE.

We now compute all SPNE. In the case of flexible sharing-rates $\sum_{n=1}^N g_{nm} \geq 2$ corresponds to the condition $P = \bigcap_{j=1}^J P_j^e = \emptyset$ in the case of fixed sharing-rates. The proof is similar to the proof of Theorem 1.

Since the planner has a non-zero corners utility function, he has positive utility only if he has access to transmit resource to all the agents. In order to prove the claims (a) and (b), we can just prove that there is no SPNE such that the group of intermediaries chosen by the planner spans all the agents and have sum of their cost in $(0, I)$. We rank the group of intermediaries with links to all the agents in non-decreasing order of the total cost of intermediaries in that group, say $P_1^e, \dots, P_J^e$. These are the efficient groups. Assume the total cost of group P_j^e is $TC_j(c) = \sum_{C_n \in P_j^e} c_n$. Suppose $TC_1(c) \in (0, I)$.

If $TC_1(c) < TC_2(c)$, then the planner chooses group P_1^e. Thus, intermediary $C_n \in P_1^e$ has the incentive to increase cost c_n to get higher payment from planner as long as group P_1^e is still the group with the lowest total cost. Thus, this cannot be a SPNE.

If $TC_1(c) = TC_2(c) = \cdots = TC_k(c) < TC_{k+1}(c)$, assume $P' = \cap_{j=1}^k P_j^e$. When $P' \neq \emptyset$, the intermediary $C_n \in P'$ has the incentive to increase the cost c_n, as long as group P_1^e is still the group with lowest total cost, thus this cannot be a SPNE. Alternatively, when $P' = \emptyset$, assume planner selects group P_j^e, then $\exists$ intermediary $C_i \in P_{j'}^e$, $C_i \notin P_j^e$ and $c_i > 0$, $1 \leq j, j' \leq k$. Since $TC_1(c) \in (0, I)$, intermediary C_i has the incentive to lower its cost to be selected by the planner. So there is no SPNE with $TC_1(c) \in (0, I)$.

Finally, note that there might be an inefficient equilibrium where the planner pays not less than I. This happens when $TC_1(c) \geq I$, and no intermediary C_i can lower its cost to achieve a total cost less than I, as shown in example 18.

2. Assume the $g_{11} = 1$, and $g_{n1} = 0$ $\forall n \geq 2$. There is only one link to agent 1. Since the planner has non-zero corners utility functions, utility is 0 if no resource is not transmitted to agent 1. When intermediary C_1 charges $c_1 < I$, the planner will pay the cost and transmit the rest of resource to achieve positive utility. Thus the intermediary C_1 charges I in equilibrium, and planner transmits no resource to agents.

3. If there exists efficient SPNE, and $\exists m$ with $d^1(m) > d^2(m)$. From Theorem 1, then $P = \emptyset$. Assume intermediary C_n with waste rate d_n has $d^1(m) = d_n g_{nm}$. Since the utility function is non-zero corners, the planner will use some intermediary to transmit resource to agent m. The planner will choose the intermediaries C_k with $d^1(m') = d_k g_{km'}$, for m' in efficient group of intermediaries. Since C_n is the only most efficient intermediary linked to m, C_n is in every efficient group, which means $C_n \in P \neq \emptyset$, thus there is efficient SPNE only if $d^1(m) = d^2(m)$, $\forall m$.

If $d^1(m) = d^2(m)$, $\forall m$, suppose $P \neq \emptyset$, and $C_n \in P$. Assume $d_n g_{nm} = d^1(m)$ for $m = 1, \dots, \bar{m}$. There exists intermediary $C_{n(m)}$ rather than C_n with $d_{n(m)} g_{n(m)m} = d^2(m) = d^1(m)$. The group of intermediary $C(m) = \{C_{n(1)}, \dots, C_{n(\bar{m})}\}$. Then pick an efficient group P_j^e, the group of intermediaries $P_1 = P_j^e \cup C(m) \backslash C_n$ is an efficient group since it includes intermediaries with waste rates equal to $d^1(m)$, $\forall m$. $C_n \notin P_1$, contradicts with $C_n \in P$. Thus, $P = \emptyset$, there is efficient

SPNE. ■

Corollary 4 (Comparative statics on the addition of an intermediary)
Suppose the utility function of planner has non-zero corners. The intermediary C_{N+1} is added into the network with waste rate d_{N+1}, then

1. *If $\forall m$, $d_{N+1}g_{(N+1)m} \leq d^1(m)$[18], and condition (A1), (A2) satisfy, then multiplicity of equilibria might occur in the extended problem. Every robust SPNE will not be worse off than the previous robust SPNE.*

2. *If $\exists m$, $d_{N+1}g_{(N+1)m} > d^1(m)$, then the planner is not necessarily better off due to multiplicity of equilibria.*

Proof.

1. When $d_{N+1}g_{(N+1)m} < d^1(m)$, $\forall m$, C_{N+1} does not belong to any new efficient group P^e_{1j}, the efficient groups are the same, $P_0 = P_1 = \{C_1, \ldots, C_k\}$, then $c'_j = 0$ for $j \geq k$ in a robust SPNE. The maximal utility $\bar{u}$ of planner will not change. Condition (A1), (A2) satisfy, there is unique robust SPNE, the cost allocation satisfy (A1) $(c_1, \ldots, c_k)$ solves the equations $(I - \sum_{n=1}^{k} c_n)\bar{u} = (I - \sum_{n=1,n\neq j}^{k} c_n)\bar{u}_{-j}$, thus $c_j = (I - \sum_{n=1}^{k} c_n)(\frac{\bar{u}}{\bar{u}_{-j}} - 1)$. For $j \leq k$, because the efficient group would be used in SPNE and its utility will be no less than group without intermediary C_j, $(I - \sum_{n=1}^{k} c'_n)\bar{u} \geq (I - \sum_{n=1,n\neq j}^{k} c'_n)\bar{u}'_{-j}$, thus $c'_j \leq (I - \sum_{n=1}^{k} c'_n)(\frac{\bar{u}}{\bar{u}'_{-j}} - 1)$. By definition, we will have $\bar{u}'_{-j} \geq \bar{u}_{-j}$. If $\sum_{n=1}^{k} c'_n > \sum_{n=1}^{k} c_n$, $\exists c'_j > c_j$, there is $I - \sum_{n=1}^{k} c_n > I - \sum_{n=1}^{k} c'_n$, then $c'_j > c_j = (I - \sum_{n=1}^{k} c_n)(\frac{\bar{u}}{\bar{u}_{-j}} - 1) > (I - \sum_{n=1}^{k} c'_n)(\frac{\bar{u}}{\bar{u}'_{-j}} - 1)$, contradiction. So total cost $\sum_{n=1}^{k} c'_n \leq \sum_{n=1}^{k} c_n$, the robust SPNE will not be worse off than previous robust SPNE.

When $d_{N+1}g_{(N+1)m} \leq d^1(m)$, $\forall m$ and $d_{N+1}g_{(N+1)m} = d^1(m)$ for some m, then $C_{N+1} \in P^e_{1j}$, $C_{N+1} \notin P_1$. In this case, $P_1 \subseteq P_0$, then $c'_j = 0$ for $j \geq k$ in a robust SPNE. The proof of the first case follows. The proof is similar to part 1 and 2 of Corollary 3.

2. When $\exists m$, $d_{N+1}g_{(N+1)m} > d^1(m)$, it means $C_{N+1} \in P_1$, the new intermediary would charge positive cost. The robust SPNE may be very different with the robust SPNE before adding C_{N+1}. At the same time, multiple robust SPNE which result in different utility for planner may happen in this case (this occur in example 17). Due to the multiplicity of equilibria, the planner is not necessarily better-off. ■

7 Conclusion

This paper investigates how intermediation affects the resource transmission between a planner and agents. We built a game theory model to study the market power of intermediaries to charge planner cost for using their links to transmit resources to the agents. We discover and describe the necessary and sufficient conditions for the efficient SPNE and uniqueness of an efficient robust SPNE. Properties of the SPNEs, including inefficient SPNE, are provided. Conditions for uniqueness of robust SPNE are also discussed. For the comparative static analysis, we demonstrate how to add intermediaries to achieve an efficient equilibrium, and study the effect of adding one intermediary. Under some condition, adding one intermediary will result in higher utility for the planner, but this result may fail due to the multiplicity of equilibria.

[18] $d^1(m)$ is the highest waste rate linked to agent m before adding intermediary C_{N+1}, similar to $d^2(m)$. The rank is among intermediaries $\{C_1, \ldots, C_N\}$

This paper is a start in the analysis of resource transmission from a planner to agents via intermediaries. We anticipate that this paper will be complemented by follow-ups, including studying the case of (competition between) multiple planners, variable-cost setting instead of fixed-cost setting by the intermediaries, as well as the study of multiple layers of intermediation.

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