Optimal Auctions with Ex-Post Verification and Limited Punishments

Volodymyr Baranovskyi and Sergii Golovko *

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Abstract

In this paper, we consider an auction environment in which after the sale, the seller has the opportunity to verify the winner's ex-post value and impose a limited punishment for "underbidding." Investigating how the seller should approach this opportunity, we show that even small penalties allow the seller to significantly increase her revenue. In our environment, the first-price auction with an optimally chosen penalty rule is optimal among all winner-pay auctions. Before the auction begins, the seller recommends a bidding strategy to the bidders. If the auction winner bids at least as much as the seller has suggested, the winner is not punished; if, on the other hand, the winner does not bid as much as has been recommended, he is punished, with the penalty increasing as the buyer deviates more and more from the recommendation.

Our results indicate several qualitative differences from standard (without ex-post punishments) auctions. In equilibrium, buyers bid more aggressively; the optimal reserve price is lower; and the revenue-equivalence principle does not hold—we state conditions under which a first-price auction is superior to a second-price auction. Our results also lead us to suggest the following recommendation for policymakers: A government may increase its revenue when auctioning publicly owned assets by providing tax concessions to buyers who submit sufficiently high bids.

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1 Introduction

During the last few decades, auctions have been extensively used by governments for privatization. Examples include the sale of large enterprises in eastern and western Europe, the licensing of the electromagnetic spectrum in the United States, and the sale of construction contracts. In each of these auctions, the relationship between the seller and the buyer extends beyond the auction setting, as a privatized firm pays taxes, is monitored by the government, and is affected by ongoing industry regulations. As a result, the government may gain access to information regarding the buyer's true valuation. If the government realizes ex-post that the price paid in the auction was too low, it can use its regulatory power to penalize the buyer.

For example, in 2004, the Ukrainian government sold a major steel producer, Kryvorizhstal, to a consortium led by the president's son-in-law for \$800 million. This price was considered to be too low, and the incident was used as a stunning example of corruption during the following presidential election. The new government deemed the instance of privatization illegal and canceled the sale, and Kryvorizhstal was eventually resold for \$4.81 billion. The New York Times called the deal "Ukraine's biggest and most profitable privatization auction."[1]

This paper addresses situations in which the seller's ability to punish bidders is limited. For instance, in repeated procurement auctions, if the government receives paperwork from a bidder with a history of previous negative interactions, it may disqualify the bidder. Our results indicate that even if the probability of this occurring is quite small, its effect on the seller's expected revenue is significant. Therefore, knowing how to construct an optimal mechanism in environments with ex-post verification and limited punishments may be quite important in practice.

In our model, a profit-seeking seller uses an auction to allocate an indivisible object to one of several ex-ante identical buyers. Prior to the auction, the seller commits to both an auction format and a penalty rule as a function of the winning bid and the winner's valuation. We assume that after the auction, the winner's value can be precisely verified through, for example, observing the revenue stream. Once the winner's value is established, the seller is able to impose a punishment and the winner suffers a corresponding loss. Only a part of this punishment contributes to the seller's revenue, however. In this way, we incorporate both monetary and non-

[1] The New York Times, March 15th, 2006.

monetary punishment possibilities, as well as the costliness of imposing a punishment.

In this paper, when considering winner-pay auctions, we find the optimal selling mechanism, which includes both an auction format and a penalty rule, and study the efficiency of an optimal auction. The latter is important in practice. If the reserve price is set too high, the government may fail to sell the object. If the price is too low, however, the power can face critics from the public once the true value of object is verified. As the Economist notes in discussing the privatization of the Royal Mail though the IPO: "The Royal Mail sale was a reminder of the political risks: price an asset too high and the deal might flop; price it too low and the taxpayer feels cheated." [2]

We start our analysis by studying a first-price auction. Our first observation, which extends to other auction formats as well, is that the problem of finding an optimal penalty rule can be reduced to the design of an optimal type-dependent recommendation. Before an auction starts, the seller announces how much he expects each type of buyer to bid. The recommendation is anonymous, as it depends on the buyer's valuation rather than his identity. If the winner follows the seller's recommendation, he is not penalized (or penalized a lesser amount); otherwise, he can be penalized the maximum amount, provided he is the winner. Our second observation is that unlike in standard auctions, equilibrium recommendations, or the ones from which bidders do not find it profitable to deviate, can be non-monotonic: that is, a bidder with a high valuation may be prescribed to bid less than a bidder with a lower valuation.

Nevertheless, we show that the seller's optimal recommendation is monotonic. The penalty is used purely as a threat—the buyer is never punished, provided he follows the recommendation. The standard methods that prescribe using differential equations to derive the symmetric equilibrium are not applied in our environment. Instead, we use the following recursive procedure to find the optimal recommendation for a first-price auction. We divide the type space into finitely many intervals. If the buyer's type falls in the very left interval, the buyer is prescribed to bid his value. If the buyer's type is in the next interval, then the buyer's bid is determined in order to make him indifferent between bidding according to the recommendation without being punished and mimicking one of the buyer's types in the previous interval while being punished at the maximum amount. Similarly, for each consequent interval, the

[2] The Economist, January 9th 2014.

buyer is indifferent between following the recommendation and mimicking one of the buyer's types from all previous intervals at the cost of being punished. Using the recursive structure of equilibrium, we show that buyers bid more in this context than in a standard (without ex-post penalties) first-price auction.

We further investigate how the efficiency of a first-price auction compares to that of a standard first-price auction. In our setting, the optimal reserve price is lower than it is in a standard environment. In a standard first-price auction, lowering the reserve price has two effects on the seller's revenue. First, a larger set of bidder types submit bids above the reserve price, which increases the probability of selling the object. Second, buyers adjust their behaviour by bidding less aggressively, which lowers the price, conditional on a sale. At the optimal reserve price, these two effects offset each other. In our model, the same effects are in play. However, the second effect is smaller, and it is completely absent for buyer types that are just above the reserve price (for these types, the seller effectively imposes penalties in order to extract the full surplus). Hence, the total seller's benefits from reducing the reserve price are greater than they are in the standard environment, and the two effects equalize at a lower reserve price.

By focusing our attention on monotonic recommendations, we are able to show that the first-price auction is the optimal auction format among all winner-pay auctions. For any other auction format, the seller is able to raise the same revenue by using a first-price auction instead and recommending the buyer to bid the amount a buyer of the same type would pay in expectation, conditional on winning in that auction format. The reverse of this—that there exists a monotonic bidding function of the other auction format that guarantees the same expected payment as in a first-price auction—is not always true. Perhaps surprisingly, it does not hold in a second-price auction.

While we derive the general conditions necessary for revenue equivalence to hold in the main text, a simple example of the discrete version of our model illustrates its failure. A single object is sold to one of two buyers. The values are drawn independently from the uniform distribution $V = \{2, 4, 4.9\}$. The tie is resolved by a flip of a fair coin. The seller can punish up to 60% of the value,[3] the penalty is non-monetary, and the penalty does not benefit the seller. If the object is sold through a first-price auction, the seller can extract the full surplus by recommending

[3] All our arguments go through if the seller can punish by a greater percentage than 60.

that each buyer bids his value. Provided his opponent follows this recommendation, a buyer does not have an incentive to deviate: If he wins the object by bidding differently from his value, then he is punished by 60% of his value that when added to 2 (the minimum bid required to be a winner), is larger than his value. Suppose now that the object is sold through a second-price auction. If there is a monotonic recommendation that allows the seller to extract the full surplus, then the buyer with the lowest valuation 2 should bid 2, and the buyer with valuation 4 should be prescribed to bid 8 (with a probability of 1/3 that she wins against an opponent with a valuation of 2 and with a probability of $1/2 \cdot 1/3 = 1/6$ that she wins against an opponent with a valuation of 4, making the expected payment, conditional on winning, equal to $(1/3 \cdot 2 + 1/6 \cdot 8)/(1/3 + 1/6) = 4$). However, the bidder with a valuation of 4.9 never finds it optimal to bid more than 8, since in that case his expected payment, conditional on winning, is greater than $(1/3 \cdot 2 + 1/3 \cdot 8)/(1/3 + 1/3) = 5$. We thus conclude that there is no symmetric monotonic equilibrium that allows the seller to extract the full surplus.

Finally, we apply the theory developed in this paper to analyze a problem of optimal taxation within a newly privatized firm. We limit taxation policies to the ones in which the future firm's profits are taxed at a flat rate that does not exceed a certain bound. We view this bound as one that is determined by law and cannot be increased. Since taxation plays the role of a limited punishment in our model, our results indicate that it should be used only as a threat to induce buyers to bid more. As a consequence, the government gains by credibly promising to soften taxation in the case of "fair" bidding.

RELATED LITERATURE.—In a special case of our model, when the punishments are non-monetary and are not part of the seller's revenue, our structure of verification and punishment is the same as in Mylovanov and Zapechelnyuk (2014). They study an allocation problem in an environment without monetary transfers and in which punishments are the only instruments that allow a principal to elicit truthful information from agents. As a result, the optimal allocation rule is stochastic—the agents are shortlisted with probabilities that depend on each agent type, and the object is randomly allocated to a person from the shortlist. They present the surprising finding that the principal benefits from restricting participation. In contrast, in our model, monetary transfers are allowed, and the optimal mechanism is deterministic—

an object is allocated to the buyer who assigns the greatest value to it. Therefore, likewise in standard auctions, the seller benefits from more buyers participating in the auction.

In another special case of our model, when the penalty is a pure monetary transfer to the seller, our auctions are closely related to those in contingent payments literature. McAfee and McMillan (1986), in the context of procurement auctions, study linear contracts in the submitted bid and ex-post realized cost of a project, where this cost is subject to the moral hazard problem. Despite the latter, they find that an optimal linear contact is contingent on observed costs and brings in more revenue than a standard first-price auction.

In the same spirit, Hansen (1985) shows that if a seller uses auctions in which bids are made in stock or profit shares, she receives a higher expected payoff than in cash auctions. In our model, the punishment is contingent on the winner's value, and hence, as in the aforementioned two papers, the seller's revenue is larger than it is in a standard auction. In his comment to Hansen (1985), Crémer (1987) goes further and argues that if a seller is allowed to fully compensate a winner in cash, he can achieve full control of the merging firm and extract the winner's full valuation. A similar result holds in our model when the seller can apply a sufficiently high punishment. DeMarzo et al. (2005) [4] points out that if the seller is cash-constrained or initial investment is not fully contractible then Crémer's argument is not valid. They show that the seller's expected revenue is increasing in steepness of securities and maximized for call options. Our penalty design looks similar to a call, but with its exercise value increasing in winner's bid (the winner is penalized if his observed value is higher than expected based on his bid). As with classical calls in DeMarzo et al. (2005), in our setting, the first-price auction format is superior; however, here in equilibrium, it is used purely as a threat.

Our paper also relates to literature on optimal auction design. The central finding in this literature is the revenue-equivalence principle, which states that under certain conditions, two auction formats generate the same revenue. This was first observed by Vickrey (1962) in the case of first- and second-price auctions, and later generalized by Riley and Samuelson (1981) and Myerson (1981) to incorporate other auction formats as well.

[4] Skrzypacz (2013) use DeMarzo et al. (2005) as a baseline model and survey recent related literature.

The literature has identified multiple reasons for the failure of revenue equivalence by relaxing one of the assumptions in Riley and Samuelson (1981) and Myerson (1981). Thus, Milgrom and Weber (1982) study first- and second-price auctions when buyers have only partial information about their valuations. Under the assumption of interdependent values and affiliated signals, they show that a second-price auction is revenue-superior to a first-price auction. In contrast, a first-price auction is the preferable auction format when buyers are risk-averse (see, for instance, Holt Jr (1980), Matthews (1979), and Maskin and Riley (1984)). Che and Gale (1998) derive the same result under the assumption that buyers are financially constrained. In our environment, buyers bid more than their values in the second-price auction and therefore, the same ranking holds when buyers' budgets are equal to their valuations. We suspect that in our model, this result extends to include cases in which buyers have budget constraints that may exceed their valuations and, as in the work of Che and Gale (1998), budgets are buyers' private information.

Marshall and Marx (2007), by studying buyers' behaviour in cartels that are not all-inclusive, show that a second-price auction is more susceptible to collusion than a first-price auction. Their results are in sharp contrast with previous findings[5] that all-inclusive cartels can suppress all buyers' competition, and therefore, the revenue equivalence of first- and second-price auctions continues to hold. If we allow for the possibility of collusion in our model, all-inclusive cartels may successfully escape a punishment and buy the object at the lowest price in a second-price auction. The latter is not true for a first-price auction, and therefore, similarly to Marshall and Marx (2007), we find that a first-price auction is revenue-superior.

This paper also presents its own distinct reason for the failure of revenue equivalence. Unlike in a standard second-price auction, in a second-price auction studied in this paper, bidding one's own value is not the dominant strategy. Moreover, the existence of the monotonic equilibrium that provides the same revenue to the seller as in the optimal first-price auction is not guaranteed. This causes a first-price auction format to be superior whenever such monotonic equilibrium does not exist in a second-price auction.

[5] Bidders' behaviour in all-inclusive cartels for first-price auctions has been studied by McAfee and McMillan (1992), while a similar analysis for second-price auctions has been done by Graham and Marshall (1987) and Mailath and Zemsky (1991).

2 The Model

A seller owns a single indivisible object and values it at zero. She uses an auction to allocate this object to one of n potential buyers. Each of them knows his exact value for the object, which is private information. It is common knowledge that bidders' values v_i are independently and identically distributed on $V = [v_L, v_H]$ with cumulative distribution function F, whose density f is continuous and positive on V. All parties, including the seller, are expected profit-maximizers. We assume that bidders are not subject to any liquidity or budget constraints. (A violation of the latter assumption is briefly discussed in Section 4.)

We focus our attention on the winner-pay auction, or the type of auction in which the highest bidder (if two bidders are tied, the tie is resolved by the flip of a fair coin) wins the object if his offer meets a reserve price r and he is the only one who makes a payment to the seller.[6] Each winner-pay auction is completely defined by its winner's payment function $M : \mathbb{R}^n_+ \to \mathbb{R}_+$. We assume that M satisfies the following assumptions:

Assumption 1 (Anonymity). *$M(\mathbf{b})$ is invariant to permutations of $\mathbf{b}$.*

Assumption 2 (Continuity). *$M(\mathbf{b})$ is continuous.*

Assumption 3 (Monotonicity). *For all $\mathbf{b}, \mathbf{b}' \in \mathbb{R}^n_+$, $M(\mathbf{b}) \geq M(\mathbf{b}')$ if $\mathbf{b} \geq \mathbf{b}'$ and $M(\mathbf{b}) > M(\mathbf{b}')$ if $\mathbf{b} > \mathbf{b}'$.* [7]

Assumption 4 (Reserve Price). *$M(\mathbf{b}) = r$ if $r = \max\limits_i \mathbf{b}_i$.*

Note that two major auction formats, first- and second-price sealed-bid auctions with reserve price r, satisfy Assumptions 1-4. More generally, each auction format with a payment function that is a linear combination of all submitted bids satisfies Assumptions 1-4.

Unlike in standard auctions, in our setting, a seller can impose a penalty on the winner. This penalty depends on the winner's bid and the ex-post verified value. We assume that the seller can commit to a penalty rule $\gamma(b, v) : \mathbb{R}_+ \times V \to \mathbb{R}_+$, where the first argument is a bid and the second argument is the ex-post verified

[6] The all-pay auction is the most natural example that does not belong to this class.

[7] Here, we adopt the following means of comparing two vectors $\mathbf{b}$ and $\mathbf{b}'$: $\mathbf{b} \geq \mathbf{b}'$ if and only if $\mathbf{b}_i \geq \mathbf{b}'_i$ for all i; $\mathbf{b} > \mathbf{b}'$ if and only if $\mathbf{b}_i > \mathbf{b}'_i$

winner's value. Similarly to Mylovanov and Zapechelnyuk (2014), we assume limited punishments: The winner with value v cannot be penalized more than $c(v)$. That is, for any $b \in \mathbb{R}_+ : \gamma(b, v) \leq c(v)$. We assume that this maximum penalty function $c(v)$ satisfies the following assumptions:

Assumption 5 (Continuity)**.** *$c(v)$ is continuous.*

Assumption 6 (Monotonicity of Penalty)**.** *$c(v)$ is positive and increasing.*

Assumption 7 (Monotonicity of Surplus)**.** *$v - c(v)$ is positive and increasing.*

Example 1 (Linear Penalty)**.** *The function $c(v) = av$, where $a \in (0, 1)$, satisfies Assumptions 5-7.*

Let $s \in [0, 1]$ be a useful share of the penalty that goes directly to the seller.[8] In one extreme case $s = 0$, the penalty is purely non-monetary (for instance, it may be a reputational loss), and the seller does not receive any direct benefits from it.[9] Mylovanov and Zapechelnyuk (2014) study this situation when monetary transfers are not allowed. In another extreme case $s = 1$, the penalty is purely monetary and may be interpreted as a payment, contingent on the ex-post verified winner's value v. DeMarzo et al. (2005) addresses this situation in the context of unlimited punishments and bidders who are imperfectly informed prior to the bidding stage. Intermediate values of s correspond to a combination of monetary and non-monetary penalties or underline costliness of punishment. All parameters of the model, except for the bidders' valuations, are common knowledge. To summarize, we specify the timeline of the model below.

Timeline. Prior to the auction, the seller announces both an auction format and a penalty rule γ. Next, bidders submit their bids. Given the profile of submitted bids $\mathbf{b}$, the object is allocated to the bidder with the highest bid, provided the bid meets the reserve price r. The losing bidders make no payments to the seller, and therefore, their payoffs are equal to zero. The winning bidder, say i, obtains the object, pays $M(\mathbf{b})$ to the seller, and suffers an additional loss of $\gamma(\mathbf{b}_i, \mathbf{v}_i)$, implying that his payoff equals $\mathbf{v}_i - M(\mathbf{b}) - \gamma(\mathbf{b}_i, \mathbf{v}_i)$. This means that the seller's payoff equals the sum of

[8] The majority of our results, in particular Theorem 1, continue to hold if we allow s to be negative. Our interpretation is that it may be costly for the seller to impose a non-monetary penalty on the winner.

[9] As our results indicate, there could be indirect benefits from such penalties, as buyers bid more aggressively in order to not be penalized.

the winner's transfer $M(\mathbf{b})$ and is discounted by a factor $s \in [0,1]$ penalty $\gamma(\mathbf{b}_i, \mathbf{v}_i)$: $M(\mathbf{b}) + s\gamma(\mathbf{b}_i, \mathbf{v}_i)$.

Equilibrium. Each auction format and penalty rule γ determine a game among the bidders. A strategy for a bidder i is a Lebesgue measurable function $\beta_i : V \to \mathbb{R}_+$, which describes his bid for every possible value. As is typically done in symmetric environments (see, for instance, Krishna (2009)), we restrict our attention to symmetric equilibria—in which the value of the bid depends only on the buyer's valuation and not on his identity—and refer to β simply as a strategy. The solution concept is a Bayesian-Nash Equilibrium: Given that other bidders' values are distributed according to the distribution function F and other bidders follow strategy β, the bidder with value $v \in V$ maximizes his expected payoff by submitting a bid $\beta(v)$. Let $G(b)$ be the probability that bid b wins the auction, and $m(b)$ be the expected payment of the bidder who wins with bid b. Then, function β is an equilibrium if for all $v \in V$:

$$G(\beta(v))(v - \gamma(\beta(v), v) - m(\beta(v))) \geq \sup_{b \in \mathbb{R}_+} G(b)(v - \gamma(b, v) - m(b)) \tag{1}$$

Note that both G and m depend on the equilibrium function β. Without loss of generality, we assume that the seller imposes the greatest penalty $c(v)$ if the bidder deviates from the prescribed strategy β:

$$G(\beta(v))(v - \gamma(v) - m(\beta(v))) \geq \sup_{b \in \mathbb{R}_+} G(b)(v - c(v) - m(b)) \tag{2}$$

Here, with some abuse of notation, $\gamma(v)$ denotes $\gamma(v, \beta(v))$. Therefore, we can interpret β together with γ as either a bidding recommendation or simply a recommendation from the seller. Provided that the auction winner does not follow the recommendation, the maximum penalty is imposed. Otherwise, he is punished by $\gamma(v)$. Defined in this way, γ depends on the bidding function β. Therefore, it is convenient to define γ as a part of the equilibrium:

Definition 1. *A recommendation (γ, β) is an equilibrium of the auction with ex-post verification and limited penalties if and only if* (2) *holds for all $v \in V$.*

Fixing an auction format, our goal is to find an equilibrium recommendation that maximizes the seller's revenue:

Definition 2. *Given an auction format, an equilibrium recommendation (γ, β) is an* *optimal recommendation if it maximizes the ex-ante seller's payoff among all equilibrium recommendations in this auction format.*

The remainder of the paper is organized as follows. In the next section, we study first-price auctions. We determine the seller's optimal recommendation and prove that it is essentially uniquely defined. We complete Section 3 by comparing the optimal reserve price in our environment with the one in standard auctions. In Section 4, we show that the optimal recommendation derived in Section 3 provides an upper bound on sellers' revenue in all winner-pay auctions. We also show that a second-price auction may provide a lower payoff than a first-price auction. The implications of our theory when considering the optimal taxation of a newly privatized firm are presented in Section 5. Our final remarks are in Section 6. All omitted proofs are collected in the Appendix.

3 First-Price Auctions

In this section we derive the optimal recommendation (γ^*, β^*) in a first-price auction. We begin this section by assuming that the reserve price r is fixed. We discuss the implications of allowing the seller to choose the reserve price in Subsection 3.3.

In a first-price auction, the expected payment, conditional on winning the object, is simply the winning bid. Therefore, the equilibrium condition can be written as:

$$G(\beta(v))(v - \gamma(v) - \beta(v)) \geq \sup_{b \in \mathbb{R}_+} G(b)(v - c(v) - b) \tag{3}$$

Finding the optimal recommendation is challenging because unlike in a standard auction, in our setting, the equilibrium bidding function β needs not be monotonic. Another complication is that there are equilibria in which a positive mass of types bid the same amount. The following example demonstrates both of these issues.

Example 2. *Values are uniformly distributed on* $[2, 10]$*; there are* $n = 2$ *bidders. The maximum penalty a seller can impose is* $c(v) = v/2$*. For concreteness, all penalties go directly to the seller,* $s = 1$*. There is no reserve price,* $r = 2$*. Given that* $\gamma(v) = 0$ *for all* v*, the following constitutes a symmetric equilibrium:*

$$\beta(v) = \begin{cases} 2, & \text{if } v \in [2,4] \cup [8,10] \\ 4, & \text{if } v \in [4,8] \end{cases}$$

Suppose that bidder 2 *follows the above strategy. We argue that it is optimal for bidder* 1 *to follow* β *also. Given the distribution of types and the prescribed strategy, bidder* 1 *faces his opponent's bids* 2 *and* 4 *with equal probabilities. Assume now that bidder* 1 *assigns value* $v \in [2,4] \cup [8,10]$ *to the object. If he follows the prescribed strategy and submits bid* 2*, then he wins the object with a probability of* $1/4$ *and his expected utility equals* $1/4(v-2)$*. Bidding below* 2 *is not optimal, as in this case he loses the auction. If he submits a bid* $b \in [2,4)$*, he wins the auction with a probability of* $1/2$ *but pays* $b+v/2$ *to the seller. Then his expected utility equals* $1/2(v-v/2-b) < 1/4(v-2)$*. If he submits bid* $b > 4$*, he wins the object with certainty and pays* $b+v/2$ *to the seller. In this case, his expected utility equals* $v/2 - b < 1/4(v-2)$*. Note that we skip the case of* $b = 4$*. By bidding* $b_\epsilon = 4 + \epsilon$ *for a small but positive* ϵ*, the bidder wins the object with a much higher probability (*1 *versus* $1/2$*) while paying just a little bit more. This results in a larger payoff for the bidder (formally a supremum over all* b_ϵ *in the left-hand side of* (3) *is greater than the payoff from deviating to* $b = 4$*). The case in which the bidder* 1 *assigns value* $v \in [4,8]$ *to the object could be similarly considered and hence, omitted.*

One useful observation from the above example is that it is sufficient to check only deviations to bids b that are in close proximity to the support of bids corresponding to the prescribed strategy when verifying whether this strategy constitutes an equilibrium. The following lemma states this observation formally.

Lemma 1. *(*γ*,* β*) is an equilibrium recommendation in a first-price sealed-bid auction if and only if* $\beta(v) + \gamma(v) \leq v$ *and*

$$G(\beta(v))(v - \gamma(v) - \beta(v)) \geq \sup_{v' \in V} H(v')(v - c(v) - \beta(v')), \tag{4}$$

holds for all $v \in V$*, where* $H(v')$ *is the probability that bid* $\beta(v')$ *is among the winning bids.*

Lemma 1 allows us to treat the problem of finding the optimal recommendation as one of mechanism design. $\beta(v) + \gamma(v) \leq v$ is an individual rationality constraint, while (4) describes a set of incentive compatibility constraints. Similar to Mylovanov and Zapechelnyuk (2014), the incentive constraints in our problem global rather than local: Unless $\gamma(v) = v$, a buyer never finds it optimal to deviate by mimicking the behaviour of close-by bidders' types. Therefore, the standard methods cannot be applied. The trick is in figuring out which constraints are binding.

The main result of this section is as follows:

Theorem 1. *Under the optimal seller's recommendation (γ^*, β^*_{FPA}), the bidder is never punished provided he follows the recommendation ($\gamma^*(v) = 0$), and every type of bidder is ex-ante worse off compared to in any other equilibrium. Moreover, β^*_{FPA} is strictly increasing on $[r, v_H]$ with $\beta^*_{FPA}(r) = r$ and is uniquely defined by the recursive formula:*

$$H^*(v)(v - \beta^*_{FPA}(v)) = \sup_{v' \leq v - c(v)} H^*(v')(v - c(v) - \beta^*_{FPA}(v')) \tag{5}$$

where $H^*(v) = F^{n-1}(v)$ if $v \geq r$ and 0 otherwise.

Theorem 1 states that the bidder is indifferent as to whether to follow the recommendation and avoid punishment or to choose the best deviation provided he will be punished by a maximum amount $c(v)$. Here, the best bidder's deviation is to mimic one of types $v' < v - c(v)$. To gain better intuition, consider the case of a single buyer:

Example 3. *A single buyer participates in an auction, $n = 1$. Then $\beta^*_{FPA}(v) = \min\{r + c(v), v\}$ for all $v \geq r$. There are two types of deviation worth considering: (1) The buyer bids below the reserve price and consequently loses the auction, and (2) The buyer bids $\beta^*_{FPA}(r) = r$ and wins the object at the lowest price, but is penalized by $c(v)$. For all values v such that $v - c(v) < r$, the first type of deviation is better; therefore, to be indifferent as to whether to win or to loose the auction, the bidder should bid according to $\beta^*_{FPA}(v) = v$. If $v - c(v) \geq r$, mimicking the type r of bidder is the best deviation. Hence, to make the buyer indifferent between winning the auction at price $\beta^*_{FPA}(v)$ and winning the auction at price r while being penalized by $c(v)$, the buyer should be prescribed to bid $\beta^*_{FPA}(v) = r + c(v)$.*

To see that (5) specifies a unique β^*_{FPA} (up to values at $v \leq r$), observe that the supremum in the left hand-side is taken over all values of v' that are bounded down away from v. Therefore, we may use previously found values of β^*_{FPA} (at all

$v' \le v - c(v)$) to evaluate β^*_{FPA} at v. Given that $c(v)$ is bounded away from zero by $c(v_L)$, the following algorithm defines β^*_{FPA} in finitely many steps:

The Algorithm for Finding β^*_{FPA}

1. Set $I_0 = [0, v_0]$, where $v_0 = r$, $\beta^*_{FPA}(v) = v$, and $k = 1$.
2. Set $I_k = (v_{k-1}, v_k]$, where $v_k = v_H$ if $v_H - c(v_H) < v_{k-1}$; and $v_k = w$, where w is a solution to $w - c(w) = v_{k-1}$ otherwise. For all $v \in I_k$, β^*_{FPA} is already known for all $v' < v - c(v) \le v_{k-1}$, and therefore $\beta^*_{FPA}(v)$ is uniquely defined by (5).
3. If $v_k = v_H$, terminate the algorithm; otherwise, set $k = k + 1$ and proceed to step 2.

Note that in the second step of the algorithm, w is uniquely defined, provided $v_H - c(v_H) \ge v_{k-1}$, as $v - c(v)$ is a continuous and strictly increasing function. The immediate application of the above algorithm is that larger punishments result in a higher revenue for the seller.

Corollary 1. *For the two maximum penalty functions c_1 and c_2, with $c_1(v) < c_2(v)$ for all $v \in V$, the seller receives a weakly higher revenue (and strictly higher if $v_H - c_1(v_H) > r$) in a first-price auction with the maximum penalty function c_2 than he would in one with the maximum penalty function c_1.*

We use the above algorithm to compute the optimal recommendation in Example 2:

Example 2.(Continued). *Values are uniformly distributed on $[2; 10]$, there are $n = 2$ bidders, $c(v) = v/2$, and $r = 2$. Then β^*_{FPA} is defined as:*

$$\beta^*_{FPA}(v) = \begin{cases} v, & \textit{if } v \in [2,4] \\ \frac{15v^2 - 24v - 16}{16(v-2)}, & \textit{if } v \in [4,10] \end{cases}$$

For comparison, the equilibrium bidding function in a standard first-price auction is $\beta_{FPA}(v) = (v+2)/2$. It is easy to verify that $\beta^*_{FPA}(v) > \beta_{FPA}(v)$ for all v. The seller's revenue equals $R^* \approx 7.19$, which is greater than $R^S \approx 4.67$—the revenue in a standard first-price auction.

The observation that buyers bid more aggressively than in a standard first-price auction can be easily generalized as such:

Corollary 2. *When following the optimal recommendation, bidders bid more than they do in a standard first-price auction. This results in a higher expected revenue for the seller.*

In example 2, the seller's ability to punish the bidder by half of his private value has a substantial effect on the seller's revenue. The seller extracts almost the full surplus—the revenue the seller receives if he sells the object with full information about each bidder's private valuation—which equals 7.33 in the example. We further show that even small punishments have a significant impact on the seller's revenue, which helps the seller to substantially reduce information loss, or the difference between the full surplus and the seller's revenue R^S in a standard first-price auction.

Example 4. *Values are uniformly distributed on* $[2; 10]$*, there are* $n = 2$ *bidders,* $c(v) = av$*, and* $r = 2$*. Figure 1 depicts the seller's revenue* $R^*(a)$ *as a function of parameter* a*. For instance, if* $a = 0.01$*, then the seller's revenue equals approximately* 5.14*—compensating for almost* 18% *of the information loss in a standard first-price auction. If the seller is able to punish the buyer by only one-thousandth of his value, she still regains* 5% *of the information loss.*

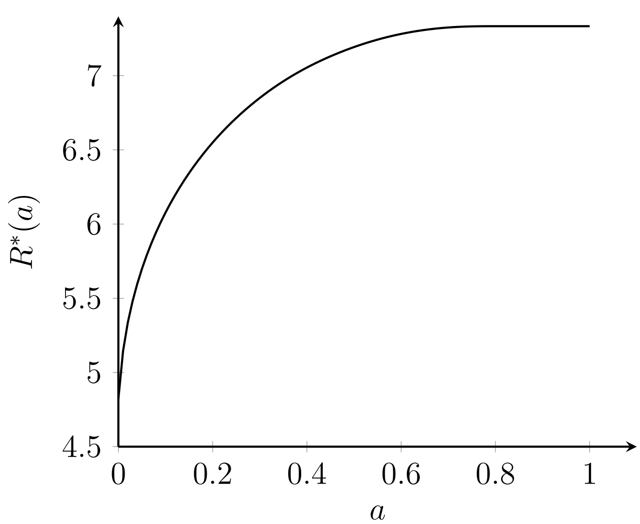

Figure 1: The Seller's Revenue in Example 4

We generalize the observation that small punishments significantly increase the seller's revenue in Proposition 1.

Proposition 1. *For a number of bidders $n \geq 2$, small linear penalties $c(v) = av$ have a large effect on the seller's revenue—that is:*

$$\lim_{a \to 0+} \frac{R^*(a) - R^S}{a} = \infty$$

In the proof we construct an equilibrium recommendation that prescribes a buyer to bid the weighted average of his value and his bid in a standard first-price auction. This recommendation is not optimal, but nevertheless, small punishments have a large impact on the seller's revenue as defined in Proposition 1. Since the optimal recommendation leads to greater revenue for the seller than a constructed recommendation would, the conclusion of Proposition 1 follows.

3.1 Proof of Theorem 1.

We divide this proof into two steps. In the first step (Proposition 2), it is proven that $(\gamma^*, \beta^*_{FPA})$ is an equilibrium recommendation in a first-price auction. To prove this, we first assume that β^*_{FPA} defined by (5) is a monotonic function. Then the probability of a buyer winning the auction with value $v \geq r$ equals $H^*(v)$. Therefore, (5) implies that all incentive constraints for which $v' \leq v - c(v)$ are satisfied. We show that when one of these constraints is binding, it is not optimal for a type v buyer to mimic a type $v' > v - c(v)$. We complete the proof of Proposition 2 by showing that β^*_{FPA} is, in fact, a monotonic function.

In the second part (the actual proof of Theorem 1), we show that $(\gamma^*, \beta^*_{FPA})$ is an optimal recommendation. For any equilibrium recommendation (γ, β), if the object is allocated to the bidder with value v, the total surplus—the sum of the seller's and the auction winner's payoff—equals $v + (s - 1)\gamma(\beta(v))$. Therefore, $(\gamma^*, \beta^*_{FPA})$ maximizes the total surplus for all possible profiles of bidders' valuations: The object is allocated to the bidder with the highest value, and he is never punished. Given that the seller's expected profit and the total surplus of the expected buyer equal the expected total surplus, it is sufficient to show that every type of buyer receives a lower expected surplus than under any other recommendation (γ, β). We show this by mathematical induction on a number of steps k of the algorithm for finding β^*_{FPA}.

Proposition 2. *$(\gamma^*, \beta^*_{FPA})$ is an equilibrium recommendation in a first-price auction.*

Proof. Assume that β^*_{FPA} is strictly increasing on $[r, v_H]$, and let $u^*(v)$ be an expected buyer's payoff given that he follows β^*_{FPA}. Then $u^*(v) = H^*(v)(v - \beta^*_{FPA})$ and equilibrium condition (4) can be rewritten as:

$$u^*(v) \geq u^*(v') + H^*(v')(v - c(v) - v') \tag{6}$$

for all $v, v' \in V$. Fix $v \in [r, v_H]$. By definition of β^*_{FPA} and u^*, (6) holds for all $v' \leq v - c(v)$. Next, observe that (5) implies that $u^*(\cdot)$ is a weakly increasing function in its argument. Then (6) also holds for all $v' \in (v - c(v), v]$. Hence, to show that β^*_{FPA} is an equilibrium, we have only left to prove that (6) holds for all $v' > v$. Suppose, by way of contradiction, that this is not true, meaning that for some $v' > v$:

$$u^*(v) < u^*(v') + H^*(v')(v - c(v) - v')$$

Let S_v be the set of all such v'. Then define $v_{inf} = \inf\{v' : v' \in S_v\}$. Take an arbitrary $v'' \in [v_{inf}, v_{inf} + c(v_L)] \cap S_v$. Since c is an increasing function, $v'' - c(v'') < v_{inf}$. Given that inequality (6) holds for all $v' < v_{inf}$ and using the definition of u^*, we have:

$$\begin{aligned} u^*(v'') - u^*(v) &= \sup_{v' < v'' - c(v'')} H^*(v')(v'' - c(v'') - \beta^*_{FPA}(v')) - \sup_{v' < v_{inf}} H^*(v')(v - c(v) - \beta^*_{FPA}(v')) \\ &\leq \sup_{v' < v_{inf}} H^*(v')(v'' - c(v'') - \beta^*_{FPA}(v')) - \sup_{v' < v_{inf}} H^*(v')(v - c(v) - \beta^*_{FPA}(v')) \\ &\leq \sup_{v' < v_{inf}} H^*(v')(v'' - c(v'') - (v - c(v))) \\ &\leq H^*(v'')(v'' - c(v'') - (v - c(v))) \leq H^*(v'')(v'' - (v - c(v))) \end{aligned}$$

where the second inequality follows from the triangle inequality for supremum and the third one follows from $v - c(v)$ be an increasing function.

Hence,

$$u^*(v) \geq u^*(v'') + H^*(v'')(v - c(v) - v'')$$

contradicting $v'' \in S_v$.

Finally, we prove that β^*_{FPA} is a strictly increasing function on $[r, v_H]$. Since $c(v) > 0$ for all $v \in V$, it is sufficient to show that $\beta^*_{FPA}(v) > \beta^*_{FPA}(w)$ for all $v > w > v - c(v)$. Using the triangle inequality for supremum once again, on the one hand,

$$\begin{aligned} u^*(v) - u^*(w) &= \sup_{v' \leq v - c(v)} H^*(v')(v - c(v) - \beta^*(v')) - \sup_{v' \leq w} H^*(v')(w - c(w) - \beta^*(v')) \\ &\leq \sup_{v' \leq w} H^*(v')(v - c(v) - \beta^*(v')) - \sup_{v' \leq w} H^*(v')(w - c(w) - \beta^*(v')) \\ &\leq \sup_{v' \leq w} H^*(v')(v - c(v) - (w - c(w))) \leq H^*(w)(v - w + c(w) - c(v)) \end{aligned}$$

On the other hand,

$$u^*(v) - u^*(w) = H^*(v)(v - \beta^*(v)) - H^*(w)(w - \beta^*(w)) \geq H^*(w)(v - w + \beta^*(w) - \beta^*(v))$$

Therefore, unless $H^*(w) = 0$,

$$\beta^*(w) \leq \beta^*(v) + c(w) - c(v) < \beta^*(v)$$

If $H^*(w) = 0$, then $u^*(v) = u^*(w) = 0$, implying that $\beta^*(v) = v > \beta^*(w)$. □

Lemma 2. *For all $w < v$,*

$$u^*(v) - u^*(w) \leq H^*(v)(v - w)$$

Proof. Given that $(\gamma^*, \beta^*_{FPA})$ is an equilibrium recommendation and using the definition of β^*_{FPA} yields:

$$u^*(v) = \sup_{v' \leq a(v)} H^*(v')(v - c(v) - \beta^*_{FPA}(v')) \tag{7}$$

for all $a(v) \geq v - c(v)$. Putting $a(v) = a(w) = v$, we have:

$$\begin{aligned} u^*(v) - u^*(w) &\leq \sup_{v' \leq v} H^*(v')(v - c(v) - \beta^*_{FPA}(v')) - \sup_{v' \leq v} H^*(v')(w - c(w) - \beta^*_{FPA}(v')) \\ &\leq \sup_{v' \leq v} H^*(v')(v - c(v) - (w - c(w))) \leq H^*(v)(v - w) \end{aligned}$$

□

Proof of Theorem 1. Let γ be an arbitrary penalty rule and β_{FPA} be a corresponding equilibrium in a first-price auction. Note that the total welfare under recommendation β^*_{FPA} is at least as large as it is under β. Therefore, it is sufficient to show that $u(v) \geq u^*(v)$ for all v, where $u(v) = G(\beta_{FPA}(v))(v - \gamma(v) - \beta_{FPA}(v))$. We prove this fact by mathematical induction on a number of steps k of the algorithm of finding β^*_{FPA}. The claim is:

Claim: For an arbitrary $k \geq 0$ and $v \leq v_k$: $u(v) \geq u^*(v)$.

Base, $k = 0$. For all $v \leq v_0 = r$: $u^*(v) = 0$ and hence, $u(v) \geq u^*(v)$.

Induction Step, $k = l + 1$. Assume that $u(v) \geq u^*(v)$ for all $v \in [0, v_l]$. Fix an arbitrary $v \in (v_l, v_{l+1}]$. Recall that by the definition of β^*_{FPA}:

$$u^*(v) = \sup_{v' \leq v - c(v)} H^*(v')(v - c(v) - \beta^*_{FPA}(v')) \tag{8}$$

Also, since β_{FPA} is an equilibrium recommendation, it should satisfy:

$$u(v) \geq \sup_b G(b)(v - c(v) - b) \tag{9}$$

Hence, it is sufficient to show that for an arbitrary $v' \leq v - c(v)$, there exists $b \in \mathbb{R}_+$ such that:

$$H^*(v')(v - c(v) - \beta^*_{FPA}(v')) \leq G(b)(v - c(v) - b) \tag{10}$$

Fix an arbitrary $v' \leq v - c(v)$. There are two possibilities: (1) $\beta^*_{FPA}(v') > \beta_{FPA}(w)$ for all $w \leq v'$ and (2) There exists $w \leq v'$ such that $\beta^*_{FPA}(v') \leq \beta_{FPA}(w)$.

Case 1. $\beta^*_{FPA}(v') > \beta_{FPA}(w)$ for all $w \leq v'$. Then define $b = \beta^*_{FPA}(v')$. This implies that by bidding b in the auction with the seller's recommendation β_{FPA}, and imposing that others bid according to a prescribed strategy β_{FPA}, a bidder wins the object in each situation in which his opponents' values do not exceed v'. Hence, $G(b) \geq H^*(v')$, which combined with $b = \beta^*_{FPA}(v')$ gives us (10).

Case 2. There exists $w \leq v'$ such that $\beta^*_{FPA}(v') \leq \beta_{FPA}(w)$. Define $b = \beta_{FPA}(w)$. Since $w \leq v_l$, by the induction hypothesis $u(w) \geq u^*(w)$, which implies

$$G(b)(w - b) \geq H^*(w)(w - \beta^*_{FPA}(w)) \tag{11}$$

Also, by Lemma 2, we have

$$H^*(w)(w-\beta^*_{FPA}(w)) \geq H^*(v')(v'-\beta^*_{FPA}(v'))-H^*(v')(v'-w) = H^*(v')(w-\beta^*_{FPA}(v')) \tag{12}$$

Combining (11) and (12) yields:

$$G(b)(w-b) \geq H^*(v')(w-\beta^*_{FPA}(v')) \tag{13}$$

From (13), it follows that $G(b) \geq H^*(v')$ as $\beta^*_{FPA}(v') \leq b$. Therefore,

$$\begin{aligned} G(b)(v-c(v)-b) &= G(b)(v-c(v)-w) + G(b)(w-b) \\ &\geq H^*(v')(v-c(v)-w) + H^*(v')(w-\beta^*_{FPA}(v')) \\ &= H^*(v')(v-c(v)-\beta^*_{FPA}(v')) \end{aligned}$$

where the inequality follows from $G(b) \geq H^*(v')$, $w \leq v-c(v)$, and (13). Hence, (10) holds. □

3.2 Uniqueness of Optimal Recommendation

Is optimal recommendation unique? According to Theorem 1, every equilibrium recommendation (γ, β_{FPA}) that provides the same ex-ante seller's payoff as the optimal recommendation $(\gamma^*, \beta^*_{FPA})$ should satisfy $u(v) = u^*(v)$ for all but the zero measure of bidder's types v. Here, as before, $u(v)$ is type v bidder's expected payoff, which corresponds to recommendation (γ, β_{FPA}), with $u^*(v)$ defined in a similar manner. In addition (γ, β_{FPA}) needs to maximize the expected total surplus: The highest-value bidder wins an object, and the sum of the penalty transfers equals zero. For $s < 1$, the latter is possible only if no penalties are imposed on the equilibrium path. Therefore, $\gamma = \gamma^*$. But then $u(v) = u^*(v)$ implies that $\beta_{FPA}(v) = \beta^*_{FPA}(v)$ for all, but positive measure of types v:

Corollary 3. *For every useful share of penalty $s \in [0, 1)$ there exists a unique optimal recommendation $(\gamma^*, \beta^*_{FPA})$.*

Before discussing the case $s = 1$, recall that we derive β^*_{FPA} based on the assumption that out of equilibrium, the behaviour of type v bidder is punished by a maximum amount $c(v)$, given that he is the winner. However, such a penalty is not necessary. In particular, if there is a single buyer, he has no incentive to bid more than prescribed, even if there is no penalty imposed. In the same vein, local deviations can be punished more deliberately: In a single-buyer case (see example 3), to prevent type v buyer, where v is such that $v - c(v) < r$, from submitting bid $v' \in [r, v]$, it is sufficient to punish such deviation by $v - v'$. This result holds more generally:

Corollary 4. *Let penalty rule $\tilde{\gamma}$ be defined as*

$$\tilde{\gamma}(b, v) = \begin{cases} 0, \ \text{if } b \geq \beta^*_{FPA}(v) \\ v - v', \ \text{if } b = \beta^*_{FPA}(v') \text{ and } v - c(v) < v' < v \\ c(v), \text{otherwise} \end{cases} \tag{14}$$

Given penalty rule $\tilde{\gamma}$, β^*_{FPA} is a symmetric equilibrium in a first-price auction.

Coming back to our discussion about the uniqueness of the optimal recommendation, if $s = 1$, we may specify a different equilibrium recommendation (γ, β_{FPA}) such that the total payment to the seller remains unchanged. To do this, define a bidding function β_{FPA} as

$$\beta_{FPA}(v) = \begin{cases} \beta^*_{FPA}(v), \ \text{if } v < \tilde{v} \\ (1 - \alpha)\beta^*_{FPA}(v) + \alpha\beta^*_{FPA}(\tilde{v}), \ \text{otherwise} \end{cases} \tag{15}$$

where $\tilde{v}$ is an arbitrary value in the interval $(v_H - c(v_H), v_H]$ and $\alpha \in (0, 1)$.

The penalty rule γ is defined, as outlined above, in order to keep the total payments to the seller unchanged: $\gamma(v) = \beta^*_{FPA}(v) - \beta_{FPA}(v)$. All deviations are punished by the maximum amount $c(v)$. To sustain (γ, β_{FPA}) as an equilibrium recommendation, it is sufficient to show that no type v bidder would aim to deviate by submitting bid $\beta_{FPA}(w)$ for some $w \geq \tilde{v}$. We show that for a sufficiently small α, the net transfers to the seller under any such deviation are greater than the net transfers to the seller under equilibrium β^*_{FPA}, provided the bidders are punished according to $\tilde{\gamma}$ defined by (15). This makes such deviations unprofitable.

Lemma 3. *For a sufficiently small but positive α recommendation, (γ, β_{FPA}), where β_{FPA} is defined according to (15) and $\gamma(v) = \beta^*_{FPA}(v) - \beta_{FPA}(v)$, constitutes an equilibrium in a first-price auction.*

Also, if there is a single buyer, the equilibrium bidding function β does not need to be monotonic. An alternative to this is presented in example 3, where the equilibrium recommendations $\beta_{FPA}(v) = r$ and $\gamma(v) = \min\{v - r, c(v)\}$ provide the same payoff $\min\{v, r + c(v)\}$ to the seller. Therefore, the uniqueness of the optimal recommendation is established up to the net transfers received by the seller, and the monotonicity of the bidding function holds if there are at least two bidders in the auction.

Corollary 5. *For a useful share of the penalty $s = 1$ and any optimal recommendation (γ, β_{FPA}), the net transfers from the winner with a value v to the seller are equal to $\beta^*_{FPA}(v)$ $(\gamma(v) + \beta_{FPA}(v) = \beta^*_{FPA}(v))$. Moreover, if the number of bidders $n \geq 2$, β_{FPA} is monotonic on $[r, v_H]$.*

3.3 Optimal Reserve Price

Suppose now that the seller sets a reserve price r in such a way as to maximize her expected revenue. Is the optimal reserve price higher or lower than the optimal reserve price in a standard first-price auction? In other words, do access to verification technology and the ability to impose limited punishments necessarily lead to improvements in auction efficiency?

To answer this question, we follow Myerson (1981) by assuming that virtual valuation $\psi(\cdot)$ that is defined as

$$\psi(v) = v - \frac{1 - F(v)}{f(v)}$$

is an increasing function (regular case). In a standard auction, the seller finds it optimal to set a reserve price equal to v_L if $\psi(v_L) \geq 0$, and $\psi^{-1}(0)$ otherwise. In the latter case, the seller excludes some bidders from the auction by setting the reserve price above their valuations.

In our setting, the following result holds:

Theorem 2. *In a regular case, the optimal reserve price is weakly lower than in a standard auction. Moreover, if $\psi(v_L) < 0$, the optimal reserve price is strictly lower than in a standard auction.*

Let r_S be the optimal reserve price in a standard first-price auction. We find that for every reserve price $r > r_S$, the seller gains at least as much as in a standard auction when reducing the reserve price by a small amount. In a regular case, such a reduction of the reserve price is beneficial for the seller in a standard first-price auction, and hence, beneficial in our environment as well. The complete proof of Theorem 2 can be found in the Appendix.

Unlike in standard auctions, deriving the explicit general formula[10] for the reserve price is possible only in very special cases.

Corollary 6. *If $c(v_H)$ is sufficiently high to make $v_H - c(v_H) \leq v_L$, it is optimal to set the reserve price equal to v_L.*

Proof. Provided $v_H - c(v_H) \leq v_L$, $\beta^*_{FPA}(v) = v$, and therefore, it is optimal to set $r = v_L$ and extract the full surplus. □

For intuition behind Theorem 2, consider a case of a single buyer—an example in which the reserve price can be found explicitly and is not necessarily equal to v_L. In this case, the problem of choosing the optimal reserve price parallels[11] the problem that the monopolist faces when selling homogeneous goods to a continuum of buyers, assuming that a measure of buyers whose valuations do not exceed v is equal to $F(v)$. Then a standard auction is essentially equivalent to a non-discrimination case: The same price p is set for all buyers. Therefore, the demand equals $q = 1 - F(p)$, and the total revenue is $TR = qp = p(1 - F(p))$. Then the marginal revenue equals

$$MR_S = \frac{dTR}{dp} \cdot \frac{dp}{dq} = p - \frac{1 - F(p)}{f(p)} = \psi(p)$$

and given that, at the optimal price $MR_S = 0$, $r_S = \psi^{-1}(0)$, as outlined above.

According to Example 3, in our environment, the monopolist has a partial ability to discriminate among buyers. To maximize her profit, she charges buyer of type v $\min\{c(v), v - p\}$ in addition to the regular price p. The total revenue then equals $TR = \int_p^{v_H} \min\{c(v) + p, v\} dF(v)$, and the marginal revenue equals

$$MR_V = \frac{dTR}{dp} \cdot \frac{dp}{dq} = p - \frac{1 - F(v_p)}{f(p)},$$

[10]Given the specific distribution function, the derivation of the optimal reserve price is straightforward.

[11]See Bulow and Roberts (1989) for a discussion of how this observation generalizes to more than one bidder in standard auctions.

where $v_p = v_H$ if $v_H - c(v_H) < p$ and is otherwise determined as a solution to $v_p - c(v_p) = p$. Since $v_p > v$, the marginal revenue MR_V is greater than the marginal revenue MR_S in a non-discrimination case. Consequently, the optimal reserve price r_V is lower than r_S.

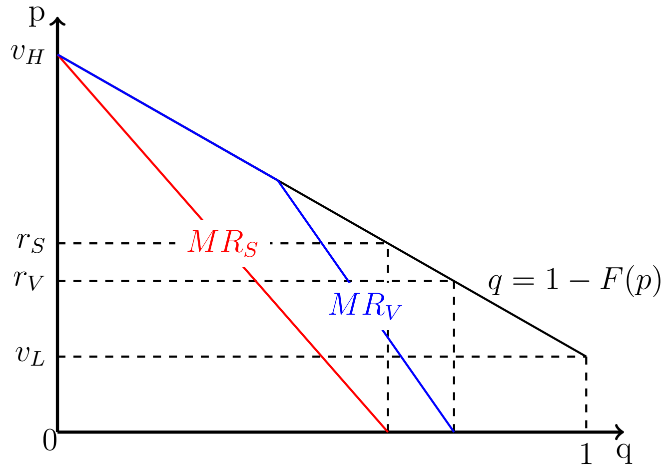

Figure 2: Comparison of Optimal Reserve Prices for a Single-Buyer Case

Why is the marginal revenue MR_V greater than MR_S? To gain further insight into this, consider a small reduction in price p (that corresponds to a small increase in quantity). The marginal benefit from this due to more buyers entering the market in both discriminatory and non-discriminatory cases equals $pf(p)$. However, the marginal losses are different in these cases. In non-discriminatory cases, all buyers above the reserve price now pay less, and the marginal loss equals $1 - F(p)$. In discriminatory cases, the buyers with valuations just above the reserve price pay their values and hence are not affected by this change. Only the buyers with valuations above v_p benefit from a reduction in prices. Therefore, the marginal loss that equals $1 - F(v_p)$ is lower than it is in non-discriminatory cases, which implies that $MR_V > MR_S$.

Define a modified virtual valuation $\phi(p)$ as

$$\phi(p) = p - \frac{1 - F(v_p)}{f(p)}$$

Provided ϕ is a monotonic function, we obtain an explicit formula for the reserve price

Proposition 3. *In a single-buyer case, if the modified virtual valuation* $\phi(\cdot)$ *is monotonic, then the optimal reserve price equals* v_L *if* $\phi(v_L) \geq 0$ *and equals* $\phi^{-1}(0)$ *otherwise.*

In the proof of Theorem 2, we show that the marginal gain from decreasing the reserve price r is at least as large as $nF^{n-1}(r)f(r)\phi(r)$. Therefore,

Corollary 7. *If the modified virtual valuation* $\phi(\cdot)$ *is monotonic, then the optimal reserve price equals* v_L *if* $\phi(v_L) \geq 0$ *and does not exceed* $\phi^{-1}(0)$ *otherwise.*

Note, that monotonicity of the modified virtual valuation holds for all convex distribution functions. In particular, if the valuations are distributed uniformly on $[v_L, v_H]$ and the penalty rule is linear $c(v) = tv$, then the optimal reserve price in a single-buyer case equals $\max\{\frac{1-t}{2-t}v_H, v_L\}$. In Example 2 ($v_L = 2$, $v_H = 10$ and $t = 0.5$), the optimal reserve price for a single-buyer case equals 10/3. For more than one buyer, it can be verified that the optimal reserve price equals 2, and hence, no buyers are excluded from the auction. For comparison, the reserve price in a standard first-price auction is 5.

Remark. *The optimal reserve price may depend on the number of bidders. An optimal auction may be efficient for number of bidders* $n \geq 2$ *even if it is not efficient in a single-bidder case.*

4 Other Auction Formats

In this section, we look beyond a first-price auction format. First we consider an arbitrary winner-pay auction. Let (γ, β) be an equilibrium recommendation. In this section, we restrict our attention to monotonic recommendations:

Definition 3. *Recommendation* (γ, β) *is a monotonic recommendation if* β *is a strictly increasing function on the interval* $[r, v_H]$.

The main result of this section is

Theorem 3. *For arbitrary monotonic recommendation* (γ, β) *of a winner-pay auction, there exists a recommendation* $(\gamma_{FPA}, \beta_{FPA})$ *of a first-price auction that provides the same ex-ante payoffs for all parties (including the seller) as* (γ, β).

We prove Theorem 3 by showing that, for a given monotonic recommendation (γ, β) of a winner-pay auction, the seller can achieve the same revenue by using the first-price auction format with seller's recommendation (γ, β_{FPA}), where $\beta_{FPA}(v) = m(\beta(v))$. That is, the seller recommends that a buyer bid the same amount he will bid in expectation, conditional on winning in the other auction format. Therefore, according to Theorem 1, the first-price auction with an optimally chosen recommendation $(\gamma^*, \beta^*_{FPA})$ provides a (weakly) greater revenue to the seller than any other winner-pay auction.

To show that other auction format may fail to provide the same amount of revenue to the seller as the optimal first-price auction format, we investigate a buyer's behaviour in a second-price auction. In a second-price auction, the winner pays the value of the second-highest bid to the seller. Let (γ, β_{SPA}) be an equilibrium recommendation in a second-price auction. Recall that without loss of generality, any violation of the equilibrium strategy β_{SPA} by a buyer with value v is punished by the maximum amount $c(v)$. In the same vein that bidding one's value is the dominant strategy in the second-price auction, the only possible deviation worth considering[12] is bidding $v - c(v)$.

Lemma 4. *(γ, β_{SPA}) is an equilibrium recommendation of the second-price auction if and only*

$$G(\beta_{SPA}(v))(v - \gamma(v) - m(\beta_{SPA}(v))) \geq G(b)(v - c(v) - m(b))$$

for $b = v - c(v)$.

Let $u(v)$ be the expected utility of type v buyer that corresponds to a symmetric equilibrium β_{SPA} in the second-price auction. As before, let $u^*(v)$ be the expected utility of type v buyer that corresponds to a symmetric equilibrium β^*_{FPA} in the optimal first-price auction. Based on Theorem 3 and Theorem 1, we may conclude that $u(v) \geq u^*(v)$ for all $v \in V$. Hence, the second-price auction is revenue-equivalent to the first-price auction with recommendation $(\gamma^*, \beta^*_{FPA})$ if $u(v) = u^*(v)$ for all but the zero measure of types v. To simplify arguments, we assume that a fraction of the penalty that goes directly to the seller $s < 1$. Therefore, $\gamma(v) = 0$ for all but the zero

[12] If a buyer with value v decides to bid differently from $\beta_{SPA}(v)$, then the buyer's net value of penalty payments is $v - c(v)$. Hence, given that he deviates, the dominant strategy is to bid $v - c(v)$.

measure of types v. To conclude:

$$F^{n-1}(v)(v - \beta^*_{FPA}(v)) = F^{n-1}(v)(v - m(\beta_{SPA}(v)))$$

for all but the zero measure of types $v > r$. Therefore, $\beta^*_{FPA}(v) = m(\beta_{SPA}(v))$ for all but the zero measure of types $v > r$.

If there is a single buyer $n = 1$, $m(\beta_{SPA}(v)) = r$ for all $v > r$. However, according to Example 3, $\beta^*_{FPA}(v) = \min\{v, r + c(v)\} > r$ for all $v > r$. Therefore, we obtain

Proposition 4. *If there is a single buyer and $s < 1$, a second-price auction provides the seller with a strictly lower payoff than the optimal first-price auction.*

Assume now that $n \geq 2$. Then given that the expected unconditional payment of type v buyer equals $rF^{n-1}(r) + \int\limits_r^v \beta_{SPA}(v)dF^{n-1}(v)$, the equilibrium condition (5) for the first-price auction can be rewritten as

$$F^{n-1}(v)v - rF^{n-1}(r) - \int\limits_r^v \beta_{SPA}(v)dF^{n-1}(v) = \sup_{v' \in V} H^*(v')(v - c(v) - m(\beta_{SPA}(v')))$$

Here as before $H^*(v') = F^{n-1}(v')$ if $v' \geq r$ and $H^*(v') = 0$ otherwise. Note that according to Lemma 4, the value of v' that maximizes the right-hand side of the above equality satisfies[13] $\beta_{SPA}(v') = v - c(v)$. Differentiating both sides of above equation with respect to v and applying the Envelope Theorem yields:

$$\beta_{SPA}(v) = v + \frac{F(v)}{(n-1)f(v)} - \frac{H^*(\beta^{-1}_{SPA}(v - c(v)))}{(n-1)F^{n-2}(v)f(v)}\left(1 - \frac{dc(v)}{dv}\right) \tag{16}$$

where we assume that $\beta_{SPA}(v) = v$ for all $v \leq r$ and $\beta^{-1}_{SPA}(w) = \sup\{v : \beta_{SPA}(v) \leq w\}$ for all w for which the inverse of β_{SPA} fails to exist. Therefore, we have proven:

Proposition 5 (Revenue Equivalence)**.** *If $s < 1$, the expected revenue in a second-price auction is the same as it is in the first-price auction (with an optimally chosen recommendation) if and only if there exists a strictly increasing bidding function β_{SPA} that satisfies (16) for all $v > r$.*

Analogous to the first-price auction, let β^*_{SPA} denote the monotonic solution of (16), provided it exists. One important feature of β^*_{SPA} is that the bid exceeds

[13] To be more rigorous, it is the maximum value of v' such that $\beta_{SPA}(v') \leq v - c(v)$.

the value[14]. We use this fact to introduce the recursive algorithm for finding β^*_{SPA} similarly to how we find that of the first-price auction. Note that this algorithm also provides a constructive way of checking the conditions of Proposition 5

The Algorithm for Finding β^*_{SPA}

1. Set $I_0 = [0, v_0]$, where $v_0 = r$; $\beta^*_{SPA}(v) = v$ and $k = 1$.

2. Set $I_k = (v_{k-1}, v_k]$, where $v_k = v_H$ if $v_H - c(v_H) < v_{k-1}$; and $v_k = w$, w is a solution to $w - c(w) = v_{k-1}$ otherwise. For all $v \in I_k$, $v - c(v) < v_{k-1}$ and therefore ${\beta^*}^{-1}_{SPA}(v - c(v))$ is well defined and hence, $\beta^*_{SPA}(v)$ is evaluated by (16).

3. If β^*_{SPA} is non-monotonic on $[0, v_k]$, then terminate the algorithm: Revenue equivalence does not hold.

4. If $v_k = v_H$, terminate the algorithm; otherwise set $k = k + 1$ and proceed to step 2.

When does revenue equivalence fail for $n \geq 2$? This question can now be rephrased as: Under what conditions does the above algorithm terminate at step 3? Assume it terminates for $k = 1$. For all $v \in I_1$, equation (16) is especially simple

$$\beta_{SPA}(v) = v + \frac{F(v)}{(n-1)f(v)} \tag{17}$$

For β_{SPA} to be non-monotonic on I_1, the density function f needs to increase sufficiently "fast" on some sub-interval of I_1 to compensate for the increase in the distribution function F and in v on the same sub-interval. We illustrate this possibility using a numerical example similar to the one presented in the Introduction for a discrete version of our model.

[14] Recall that in an optimal first-price auction buyers bid more than in a standard first-price auction. Consequently, the expected payment by type v bidder is also larger than it is in a standard auction. Provided that β^*_{SPA} and β^*_{FPA} lead to the same expected payment by type v bidder, buyers need to bid more than in a standard second-price auction (that is more than v).

Example 5. *Values are distributed on* $[2,14]$ *according to the absolutely continuous distribution function with corresponding density function*

$$f(v) = \begin{cases} \frac{1}{50}, \ \textit{if } v \in [2,11) \\ \frac{19}{100}(v-13) + \frac{2}{5}, \ \textit{if } v \in [11,13] \\ \frac{2}{5}, \ \textit{if } v \in (13,14] \end{cases}$$

There are two bidders $n = 2$*, the reserve price* $r = 2$*, and the maximum penalty function* $c(v) = 0.9v$*.*

Since $v_H - c(v_H) = 14 - 0.9 \cdot 14 < 2 = v_L$*,* $I_1 = (2,14]$*, it is possible to extract the full surplus in the first-price auction. Applying formula (17), we can compute that: (1) For all* $v < 11$*,* $\beta_{SPA}(v) = 2v - 2$*, and (2) For all* $v > 13$*,* $\beta_{SPA}(v) = v + (3/5 + 2/5(v-13))/(2/5) = 2v - 11.5$*. Therefore, for all* $v > 13$ *and* $w \in [10,11]$*,* $\beta_{SPA}(v) < \beta_{SPA}(w)$*, implying that there is no monotonic solution for (16).*

The above example can be generalized even for those cases in which the maximum penalty that the seller can impose on a buyer is sufficiently small.

Proposition 6. *For the arbitrary maximum penalty function* $c(v)$ *and* $s < 1$*, there exists an absolutely continuous distribution function* F *on* $[v_L, v_H]$ *such that the seller's revenue in a second-price auction is strictly lower than in an optimal first-price auction.*

Therefore, the failure of revenue equivalence is due to the possible non-existence of monotonic equilibrium that generates the desirable buyers expected (conditional on winning) payments. In a first-price auction, a buyer's payment is his bid and consequently, there is always a monotonic equilibrium, provided the desirable expected payment is a monotonic function in a bidder's value.

We outline two other reasons why the first-price auction format may be the preferable auction format in our environment. First, assume that buyers are financially constrained and cannot afford to pay more than their valuations. Then in a second-price auction, each buyer's bid cannot exceed his value, implying that the seller's expected revenue does not exceed that of a standard auctions. Therefore, according to Corollary 2, the first-price auction provides a strictly larger revenue to the seller.

Second, consider the possibility of collusion. For simplicity, assume that all bidders form a cartel. In a second-price auction, cartel bidders may successfully escape penalties and buy the object at a reserve price: The bidder with the highest value v bids $\beta^*_{SPA}(v)$, and the rest of the cartel members submit bids that are not serious (at or below r). In contrast, in a first-price auction, it is impossible for a cartel to buy the object at a reserve price: Even if the winner with value v wins the auction at price r, he will be punished by $c(v)$ unless $v = r$ and therefore, the total revenue equals $r + c(v)$.

5 Application: Selling a Firm by a Government.

Consider a government that wants to sell a firm via an auction. To simplify the exposition, we assume that there is no reserve price. The firm generates a revenue $R(e) = 2\sqrt{\alpha e}$, if its owner exerts an effort e and his productivity level equals α. The cost of effort is $C(e) = e$. There are n potential buyers, who are heterogeneous in their productivity levels α. Assume that productivity levels are distributed independently and identically across bidders according to the distribution function F on $[\alpha_L, \alpha_U]$. All information besides private ability levels is common knowledge. In addition to payments received during the auction stage, the government receives a tax on profit $t\pi$, where π is the profit generated by the winning bidder and $t \in (0, 1)$ is the tax rate.

Given his productivity level α, the winner maximizes the net profit of taxes: $\pi(e) = (1 - t)(2\sqrt{\alpha e} - e)$. Let v_α be the maximum value of profit the entrepreneur with productivity level α can generate. For our specific functional form, $v_\alpha = \alpha$. If we interpret $c(v) = tv$ as the maximum penalty that the government can impose on the buyers, then the current specification fits our model. There is one exception, though: The winner could generate any value $v \in (-\infty, v_\alpha]$, while in our model, we assume that the ex-post value is not under the control of the winner.

In particular, a standard auction corresponds to a situation in which penalty rule $\gamma(b, v) = c(v)$ for all values of bids b and ex-post profits v. We call this situation a flat taxation—the government imposes the same level of taxation independent of the winning bid and profit generated by the winner. Ignoring for a moment that the winner is able to generate a lower value than v_α, we know from Section 3 that flat taxation is sub-optimal. The government can do strictly better by promising to not impose any taxation if the buyer submits a bid according to the prescribed strategy. We derive a lower bound on gains from following the optimal taxation policy:

Proposition 7. *Under the optimal taxation policy, a government can gain at least*

$$n(1-t)\int_{\alpha_L}^{\alpha_U}(F^{n-1}(\alpha)-F^{n-1}((1-t)\alpha))(1-F(\alpha))d\alpha$$

compared to flat taxation.

Coming back to the possibility of a buyer generating profit below v_α, we find that it is never profitable for a buyer to plan at the bidding stage to use this opportunity after the auction is over[15]. In fact, the only possibility we need to consider is when a buyer who pretends to be of a lower type α' submits a bid $\beta(\alpha')$ and then produces α'. This results in the ex-ante utility of a type α', $u(\alpha')$ that is lower than the ex-ante utility of a type α, $u(\alpha)$. In all other cases, the deviation is detectable and therefore the maximum taxation t is imposed, making it unprofitable.

6 Conclusions

In this paper, we study an optimal auction design problem under the assumption that a buyer's private information is verified ex-post and that a seller could impose limited punishments. We have derived an optimal penalty rule for a first-price auction and have shown that small penalties have a significant impact on the seller's revenue. Before an auction begins, the seller makes recommendations regarding how much each type of buyer should bid in the auction. If the auction winner bids at least as much as the seller has prescribed, he is never punished; if he does not bid that much, the penalty increases as the buyer deviates more and more from the recommendation. The optimal recommendation is monotonic—the buyer with the highest value wins the auction, provided his bid meets a reserve price—and a buyer bids more than in a standard first-price auction. When the seller sets a reserve price, she sets it lower than in standard auctions.

We have shown that when a seller is restricted to using monotonic recommendations, a first-price auction is the optimal auction format among all winner-pay auctions. We have further proven that under certain conditions, a first-price auction is revenue superior to a second-price auction. The revenue comparison comes from the fact that a monotonic equilibrium strategy, which leads to a desirable buyers expected

[15]Note that ex-post the winner may in fact find it optimal to produce below v_α

payments, may fail to exist in a second-price auction. This finding contrasts sharply with existing auction literature, as under a wide range of assumptions, the dominant strategy for a buyer in a standard second-price auction is to bid his own value. In our environment, a buyer bids more than his value and the existence of monotonic equilibrium is not guaranteed. We conjecture that this revenue comparison holds more generally to incorporate the situations, when the seller is not restricted to using monotonic recommendations.

Our results offer, perhaps, a surprising suggestion for policymakers: A government can increase the competition when auctioning publicly owned assets by providing privileges to buyers who submit sufficiently high bids. This advice, however, should be implemented cautiously, as the exact privileges depend on the nature of information available to the government after the auction. Therefore, such policies should be used only in cases in which the primitives of the environment have been studied in-depth and are well-understood. We have illustrated one such possible application in Section 5, where we investigate an optimal taxation policy of a newly privatized firm.

We consider our paper as a first step towards understanding how monetary transfers and limited punishments work together in environments in which a seller has access to additional information regarding buyers' valuations ex-post. The natural extensions of our work include considering a model in which buyers are imperfectly informed prior to an auction or buyers' values are correlated or drawn from the different distributions. We leave these and other possible extensions for future research.

APPENDIX

Proof of Lemma 1. *Necessity.* Let (γ, β) be an equilibrium recommendation. Then (3) implies that

$$G(\beta(v))(v - \gamma(v) - \beta(v)) \geq G(b)(v - c(v) - b) \quad (18)$$

holds for all $v \in V$ and $b \in \mathbb{R}_+$. Since the above in particular holds for $b = 0$, we have $\beta(v) + \gamma(v) \leq v$. Then it is sufficient to show that for arbitrary $v, v' \in V$

$$G(\beta(v))(v - \gamma(v) - \beta(v)) \geq H(v')(v - c(v) - \beta(v')) \quad (19)$$

Fix $v, v' \in V$. If $\beta(v') \geq v - c(v)$ then, given that $G(\beta(v')) \leq H(v')$, (19) immediately follows from (18) by setting $b = \beta(v')$. Therefore, assume that $\beta(v') < v - c(v)$. For arbitrary $\epsilon > 0$ (18) yields

$$G(\beta(v))(v - \gamma(v) - \beta(v)) \geq G(\beta(v') + \epsilon)(v - c(v) - (\beta(v') + \epsilon))$$

Observe that $G(\beta(v') + \epsilon) \geq H(v')$ and therefore by taking a limit in the right-hand side of the above inequality when ϵ goes to zero we obtain (19).

Sufficiency. Assume now that (19) holds for all $v, v' \in V$ and $\beta(v) + \gamma(v) \leq v$. We want to show that (18) holds for all $v \in V$ and $b \in \mathbb{R}_+$. Fix $v \in V$ and $b \in \mathbb{R}_+$. If $v - c(v) \leq b$ then (18) trivially holds as left-hand side is non-negative.

Let $v - c(v) > b$. If $b = \beta(v')$ for some $v' \in V$ then (18) is an immediate application of (19). Assume that $b \neq \beta(v')$ and let $S = \{v' : \beta(v') > b\}$. If $S = \emptyset$, then $G(b) = 1$. Next consider a sequence $v'_{(k)}$ such that $H(v'_{(k)})$ converges to 1. Without loss of generality[16], we may assume that $\beta(v'_{(k)})$ also convergent sequence and let b' be a limit point of this sequence. Note that $b' \leq b$. For any $k \in \mathbb{N}$ (19) implies:

$$G(\beta(v))(v - \gamma(v) - \beta(v)) \geq H(v'_k)(v - c(v) - \beta(v'_k)) \quad (20)$$

Then taking a limit of the right-hand side of the above inequality and taking into account that $G(b) = 1$ and $b' < b$, (19) follows.

Suppose now that $S \neq \emptyset$ and let $b_{min} = \inf\{\beta(v') : v' \in S\}$. Then there exists a sequence $v'_{(k)} \subseteq S$ such that $\beta(v'_k)$ converges to b_{min}. As before, without loss of

[16] Otherwise, since $\beta(V) \subset [0, b]$ we may always choose a convergent subsequence.

generality, we may assume that sequence $H(v_{(k)})$ is also convergent. Let $\underline{H}$ be a limit of this sequence. Since $H(v_k) \geq G(b)$ for all k, $\underline{H} \geq G(b)$. Similarly, to the above, taking the limit of the right-hand side of inequality (20) then immediately implies (19).

□

Proof of Corollary 1. Let $(\beta^*_{FPA,i}, \gamma^*(v))$, $i \in \{1,2\}$ and $\gamma^*(v) = 0$, be the optimal recommendation in a first-price auction with the maximum penalty function c_i. If $v_H - c_1(v_H) \leq r$ then $v_H - c_2(v_H) \leq r$ and hence, $\beta^*_{FPA,i}(v) = v$ for all $v \in V$ and $i \in \{1,2\}$. Therefore, in both cases, when a maximum penalty function equals to c_1 and when it equals to c_2, the seller gets the same revenue.

Assume now that $v_H - c_1(v_H) > r$. We will do a proof by mathematical induction on the number of steps k of the algorithm for finding the optimal recommendation, to show that

Claim: For arbitrary $k \geq 1$, $\beta^*_{FPA,2}(v) \geq \beta^*_{FPA,1}(v)$ for all $v \in (r, v_k]$ with strict inequality for all $k > 1$.

Base, $k = 1$. For all $v \in (r, v_1]$: $\beta^*_{FPA,2}(v) = \beta^*_{FPA,1}(v) = v$.

Induction Step, $k = l + 1$. Let $\beta^*_{FPA,2}(v) \geq \beta^*_{FPA,1}(v)$ for all $v \in (r, v_l]$. Then for all $v \in (v_l, v_{l+1}]$ we obtain

$$\begin{aligned} H^*(v)(v - \beta^*_{FPA,2}(v)) &= \sup_{v' < v - c_2(v)} H^*(v')(v - c_2(v) - \beta^*_{FPA,2}(v')) \\ &< \sup_{v' < v - c_1(v)} H^*(v')(v - c_1(v) - \beta^*_{FPA,1}(v')) \\ &= H^*(v)(v - \beta^*_{FPA,1}(v)) \end{aligned}$$

where the inequality follows from $c_2(v) > c_1(v)$ and the induction step. Taking into account that $H^*(v) = F^{n-1}(v) > 0$, we have $\beta^*_{FPA,2}(v) > \beta^*_{FPA,1}(v)$, which ends the proof of the above claim.

Since $\beta^*_{FPA,2}(v) \geq \beta^*_{FPA,1}(v)$ for all $v \in [r, v_H]$, with a strict inequality for a positive measure of types v, the seller's revenue is higher when the maximum penalty function equals to $c_2(v)$. □

Proof of Corollary 2. Let β_{FPA} be an equilibrium bidding function in a standard auction. We will do a proof by mathematical induction on the number of steps k of the algorithm of finding β^*_{FPA}. The claim is:

Claim: For arbitrary $k \geq 1$, $\beta^*_{FPA}(v) > \beta_{FPA}(v)$ holds for all $v \in (r, v_k]$.

Base, $k = 1$**.** For all $v \in (r, v_1]$: $\beta^*_{FPA}(v) = v > \beta_{FPA}(v)$.

Induction Step, $k = l + 1$**.** Let $\beta^*_{FPA}(v) > \beta_{FPA}(v)$ for all $v \in (r, v_l]$. Fix, $v \in (v_l, v_{l+1}]$. Since $\beta_{FPA}(v)$ is a symmetric equilibrium in a standard auction, it should be the case that

$$H^*(v)(v - \beta_{FPA}(v)) \geq H^*(v')(v - \beta_{FPA}(v'))$$

for all $v' \in V$. Consequently,

$$\begin{aligned} H^*(v)(v - \beta_{FPA}(v)) &> \sup_{v' < v - c(v)} H^*(v')(v - c(v) - \beta_{FPA}(v')) \\ &\geq \sup_{v' < v - c(v)} H^*(v')(v - c(v) - \beta^*_{FPA}(v')) \\ &= H^*(v)(v - \beta^*_{FPA}(v)) \end{aligned}$$

where the second inequality follows from induction assumption. Therefore, we proved, my method of mathematical induction, that $\beta^*_{FPA}(v) > \beta_{FPA}(v)$. □

Proof of Proposition 1. Consider a recommendation $(\beta_{FPA}(v), \gamma(v))$ such that $\gamma(v) = 0$ and $\beta_{FPA}(v)$ for each v is a convex combination of a type v buyer's bid in a standard first-price auction $\beta^S_{FPA}(v)$ and his value v:

$$\beta_{FPA}(v) = (1 - k(a)) \cdot \beta^S_{FPA}(v) + k(a)v$$

where

$$k(a) = \frac{\sqrt{M^2a^2 + 4Ma} - Ma}{2}$$

and

$$M = \min_{v \in V} \frac{(n-1)vf(v)}{F(v)} > 0$$

Note, that for all $a \in [0, 1]$, $k(a) \in [0, 1)$ with $k(0) = 0$. If all the bidders follow this recommendation, then the revenue equals to $R(a) = (1 - k(a))R^S + k(a)R_{max}$, where R^S is the seller's revenue in a standard first-price auction and R_{max} denotes a value of a full surplus. Evaluating the derivative of $R(a)$ at $a = 0$ yields

$$\left.\frac{dR(a)}{a}\right|_{a=0} = \left.\frac{dk(a)}{da}\right|_{a=0} \cdot (R_{\max} - R^S) = \infty$$

If recommendation $(\beta_{FPA}(v), \gamma(v))$ is an equilibrium, then

$$\lim_{a\to 0}\frac{R^*(a) - R^S}{a} \geq \lim_{a\to 0}\frac{R(a) - R(0)}{a} = \left.\frac{dR(a)}{a}\right|_{a=0} = \infty$$

Therefore, we are left to show that the proposed recommendation $(\beta_{FPA}(v), \gamma(v))$ is an equilibrium. Since $\beta_{FPA}(v)$ is monotonic and does not exceed v, it is sufficient to check that the incentive compatibility constraints

$$F^{n-1}(v)(v - \beta_{FPA}(v)) \geq F^{n-1}(v')((1-a)v - \beta_{FPA}(v'))$$

hold for all v and v' from the interval $[r, v_H]$. Given that $\beta^S_{FPA}(v) = v - \frac{1}{F^{n-1}(v)}\int\limits_r^v F^{n-1}(t)dt$ and the way we defined β_{FPA}, the above inequality can be rewritten as

$$(1-k(a))\int\limits_r^v F^{n-1}(t)dt \geq F^{n-1}(v')((1-a)v - v') + (1-k(a))\int\limits_r^{v'} F^{n-1}(t)dt \qquad (21)$$

For fixed $v \in [r, v_H]$ and $a \in [0, 1]$ we need only to check that the above inequality holds for v' at which the right-hand side achieves its maximum on V (clearly, if inequality (21) holds for all $v' \in V$, it also holds for all $v' \in [r, v_H]$). Provided that v' can take values from the closed interval $[v_L, v_H]$ and that the right-hand side is continuously differentiable, the maximum is attained at either end-points—v_L and v_H—or at values of v' at which the derivative of the right-hand side equals to 0. Note that inequality (21) is clearly satisfied at $v' = v_L$ as the left-hand side is non-negative and the right-hand side is non-positive at $v' = v_L$. Differentiate the right-hand side of inequality (21) with respect to v', we obtain

$$\frac{dRHS(v')}{dv'} = (n-1)F^{n-2}(v')f(v')((1-a)v - v') - F^{n-1}(v') + (1-k(a))F^{n-1}(v')$$

For all $v' > (1-a)v$, the derivative of $RHS(v')$ is negative and therefore, the maximum is never attained at $v' = v_H$. Therefore, we left with verifying that inequality (21) holds for all $v' = v^*$ at which the derivative of $RHS(v')$ equals to 0:

$$v^* + \frac{k(a)F(v^*)}{(n-1)f(v^*)} = (1-a)v$$

Substituting the expression of v^* into (21) and rearranging terms we have

$$(1-k(a)) \int_{v^*}^{v} F^{n-1}(t)dt \geq \frac{k(a)F^n(v^*)}{(n-1)f(v^*)} \tag{22}$$

Next observe that the left-hand side of the last inequality satisfies

$$\begin{aligned}(1-k(a)) \int_{v^*}^{v} F^{n-1}(t)dt &\geq (1-k(a))F^{n-1}(v^*)(v-v^*) \\ &\geq (1-k(a))F^{n-1}(v^*)\left(av + \frac{k(a)F(v^*)}{(n-1)f(v^*)}\right)\end{aligned}$$

Therefore, to show that (22) holds, it is sufficient to check that

$$(1-k(a))\left(av + \frac{k(a)F(v^*)}{(n-1)f(v^*)}\right) \geq \frac{k(a)F^(v^*)}{(n-1)f(v^*)}$$

or what is equivalent

$$(1-k(a))av \geq k^2(a)\frac{F(v^*)}{(n-1)f(v^*)}$$

The latest is true as

$$\frac{k^2(a)}{1-k(a)} = aM = a \cdot \min_{v \in V} \frac{(n-1)vf(v)}{F(v)} \leq a \cdot \frac{(n-1)v^*f(v^*)}{F(v^*)}$$

□

Proof of Corollary 4. Since $\tilde{\gamma}(b,v) = \gamma^*(b,v)$ for all $b < \beta^*_{FPA}(v - c(v))$, we need only to insure that equilibrium condition (1) holds for all $b \geq \beta^*_{FPA}(v - c(v))$. Following the logic of Lemma 2 we consider only b for which there exists v' such that $\beta^*_{FPA}(v') = b$. Then for all $v' \in [v - c(v), v]$ using monotonicity of u^* yields

$$\begin{aligned}u^*(v) \geq u^*(v') &= H^*(v')(v - (v - v') - \beta^*_{FPA}(v)) \\ &= H^*(v')(v - \tilde{\gamma}(\beta^*_{FPA}(v'), v') - \beta^*_{FPA}(v))\end{aligned}$$

For $v' > v$, using Lemma 2 we obtain

$$u^*(v) \geq u^*(v') - H^*(v')(v' - v) = H^*(v')(v - \tilde{\gamma}(\beta^*_{FPA}(v'), v) - \beta^*_{FPA}(v'))$$

Therefore, β^*_{FPA} is an equilibrium induced by penalty rule $\tilde{\gamma}$.

□

Proof of Lemma 3. Let α be small enough positive number such that $\beta^*_{FPA}(v) - \beta_{FPA}(v) \leq \lambda = \min\{c(\tilde{v}), \tilde{v} - (v_H - c(v_H))\}$ for all $v \in [\tilde{v}, v_H]$. Given that $\beta_{FPA}(v)$ coincides with $\beta^*_{FPA}(v)$ for all $v \leq \tilde{v}$, we need only to check that equilibrium condition (6) holds for all $v \in V$ and $v' \geq \tilde{v}$. Provided our choice of λ for all $v \in V$ and $v' \geq \tilde{v}$

$$\beta^*_{FPA}(v) + \tilde{\gamma}(\beta^*_{FPA}(v'), v) \leq \beta_{FPA}(v) + c(v)$$

where $\tilde{\gamma}$ is defined in Corollary 4. According to Corollary 4, β^*_{FPA} is an equilibrium in the first-price auction with penalty rule $\tilde{\gamma}$. Therefore,

$$u^*(v) \geq H^*(v')(v - \tilde{\gamma}(\beta^*_{FPA}(v'), v) - \beta^*_{FPA}(v')) \geq H^*(v')(v - c(v) - \beta^*_{FPA}(v'))$$

To complete the proof recall that by construction of β_{FPA}, the expected utility of bidder with type v is the same under equilibrium recommendation (γ, β_{FPA}) and $(\gamma^*, \beta^*_{FPA})$. □

Proof of Theorem 2. With some abuse of notation, let $\beta^*_{FPA}(v, r)$ denotes the amount a buyer with value v bids under the optimal recommendation, provided the reserve price equals to r. For concreteness, assume that $\beta^*_{FPA}(v, r) = v$ if $v < r$. We first derive a lower bound for expected gains in the seller's profit when the reserve price is reduced by a small amount $\Delta r > 0$. We do this by showing that for all $v > v_L$

$$\beta^*_{FPA}(v, r) - \beta^*_{FPA}(v, r - \Delta r) \leq \frac{\Delta r F^{n-1}(r)}{F^{n-1}(v)} \tag{23}$$

(23) is proved by mathematical induction on the number of steps k of the algorithm of finding β^*_{FPA} when reserve price is $r - \Delta r$. The claim is:

Claim: For arbitrary $k \geq 1$ (23) holds for all $v \leq v_k$.

Base, $k = 1$**.** For all $v \leq v_1 = v_{r-\Delta r}$: $\beta^*_{FPA}(v, r - \Delta r) = \beta^*_{FPA}(v, r) = v$.

Induction Step, $k = l + 1$**.** Let (23) holds for all $v \leq v_l$. For arbitrary $v \in$

$(v_l, v_{l+1}]$ we may write the equilibrium condition as

$$F^{n-1}(v)(v-\beta^*_{FPA}(v,r-\Delta r)) = \sup_{v'<v-c(v)} F^{n-1}(v')(v-c(v)-\beta^*_{FPA}(v',r-\Delta r))$$

We derive an upper bound on the right-had side of the above equality, by splitting supremum in the left-hand side into two parts: 1) for all $v' \leq r$ and 2) for all $v' > r$. For the first part we obtain:

$$\begin{aligned}\sup_{v'\leq r} F^{n-1}(v')(v-c(v)-\beta^*_{FPA}(v',r-\Delta r)) &\leq F^{n-1}(r)(v-c(v)-(r-\Delta r))\\ &= F^{n-1}(r)(v-c(v)-r)+\Delta r F^{n-1}(r)\\ &\leq F^{n-1}(v)(v-\beta^*_{FPA}(v,r))+\Delta r F^{n-1}(r)\end{aligned}$$

For the second part we obtain:

$$\begin{aligned}&\sup_{v'\in(r,v-c(v)]} F^{n-1}(v')(v-c(v)-\beta^*_{FPA}(v',r-\Delta r))\\ &\quad\leq \sup_{v'\in(r,v-c(v)]} F^{n-1}(v')(v-c(v)-\beta^*_{FPA}(v',r))+\Delta r F^{n-1}(r)\\ &\quad\leq F^{n-1}(v)(v-\beta^*_{FPA}(v,r))+\Delta r F^{n-1}(r)\end{aligned}$$

where the first inequality follows from induction assumption as $v' \leq v-c(v) \leq v_l$ and the second one is an equilibrium condition when reserve price equals to r.

Combining two parts together we have:

$$F^{n-1}(v)(v-\beta^*_{FPA}(v,r-\Delta r)) \leq F^{n-1}(v)(v-\beta^*_{FPA}(v,r))+\Delta r F^{n-1}(r)$$

Dividing both sides of above inequality by $F^{n-1}(v)$ and rearranging the terms result in (23). Hence, by method of mathematical induction (23) holds for all $v > v_L$.

Let v be the highest value among the bidders. Then by reducing the reserve price by Δr, the seller gains by making a sale at price v whenever $v \in [r-\Delta r, r]$, is indifferent whenever $v \in [r, v_{r-\Delta r}]$, and suffers a loss that is bounded by expression in the right-hand side of (23) whenever $v > v_{r-\Delta r}$. Therefore, the expected gain from reducing the reserve price by Δr is at least as large as:

$$LB(\Delta r) = \int\limits_{r-\Delta r}^{r} vdF^n(v) - \int\limits_{v_{r-\Delta r}}^{v_H} \frac{\Delta r F^{n-1}(r)}{F^{n-1}(v)} dF^n(v)$$

which after simplification results in

$$LB(\Delta r) = \int\limits_{r-\Delta r}^{r} vdF^n(v) - n\Delta r F^{n-1}(r)(1 - F(v_{r-\Delta r}))$$

Taking the limit from the last expression when Δr goes to zero, results in:

$$\lim_{\Delta r \to 0} LB(\Delta r) = nF^{n-1}(r)f(r)\left(r - \frac{1 - F(v_r)}{f(r)}\right)$$

Note that the expression in the brackets is strictly larger than the virtual valuation $\psi(r)$. Hence, whenever the reserve price r exceeds the optimal reserve price r_S in standard auctions, the seller finds it profitable to reduce the reserve price by a small amount. If $r_S > v_L$ then at $r = r_S$, unlike the standard auctions, the gains are still positive and therefore the seller benefits from reducing the reserve price below r_S. Finally, the existence of optimal reserve price in our environment follows directly from continuity of $\beta^*(v, r)$ in the second argument, which is an immediate application of (23).

□

Proof of Theorem 3. Let (γ, β) be an optimal monotonic recommendation in some winner-pay auction. Then

$$H^*(v)(v - \gamma(v) - m(\beta(v))) \geq \sup_{v' \in v} H^*(v')(v - c(v) - m(\beta(v')))$$

Consider now a recommendation (γ, β_{FPA}), where $\beta_{FPA}(v) = m(\beta(v))$. Provided that $m(\beta(\cdot))$ is a monotonic function Lemma 2 guarantees that (γ, β_{FPA}) is an equilibrium recommendation in the first-price auction. Therefore, we complete the proof by showing that

Lemma 5. *Let $\beta(\cdot)$ be strictly increasing function on $[r, v_H]$. Then $m(\beta(\cdot))$ is also a strictly increasing function on $[r, v_H]$.*

Proof. Fix arbitrary $v, v' \in [r, v_H]$ such that $v' > v$. For any $w \in \mathbb{R}_+^{n-1}$, let $y_k(w)$ be a k-dimensional vector with coordinates equal to the first k coordinates of vector w, and $z_k(w)$ be a $n-1$-dimensional vector, first k coordinates of which coincides with first k coordinates of w and each of the last $n-k-1$ coordinates equals to v. Also for arbitrary k-dimensional vector x, let $\mathbb{F}(x) = \prod_{i=1}^{k} F(x_i)$. Then using monotonicity of M, for all natural $k \leq n-1$ we have

$$F^{n-1}(v)m(\beta(v)) = \int\limits_{\{w \in V^{n-1}: w_i < v\}} M(v, w) d\mathbb{F}(w) \leq F^{n-1-k}(v) \int\limits_{\{w \in V^{n-1}: w_i < v\}} M(v, z_k(w)) d\mathbb{F}(y_k(w))$$

which implies that

$$F^k(w)m(\beta(v)) \leq \int\limits_{\{w \in V^{n-1}: w_i < v\}} M(v, z_k(w)) d\mathbb{F}(y_k(w)), \tag{24}$$

Note, that above also holds for $k = 0$ if with some abuse of notation we assume that $M(v, z_0(w)) = \int\limits_{\{w \in V^{n-1}: w_i < v\}} M(v, z_0(w)) d\mathbb{F}(y_0(w))$. Finally,

$$\begin{aligned} F^{n-1}(v')m(\beta(v')) &= \int\limits_{\{w \in V^{n-1}: w_i < v'\}} M(v, w) d\mathbb{F}(w) \\ &\geq \sum_{k=0}^{n-1} C_n^k [F(v') - F(v)]^{n-1-k} \int\limits_{\{w \in V^{n-1}: w_i < v\}} M(v, z_k(w)) d\mathbb{F}(y_k(w)) \\ &\geq \sum_{k=0}^{n-1} C_n^k [F(v') - F(v)]^{n-1-k} F^k(v) m(\beta(v)) = F^{n-1}(v') m(\beta(v)), \end{aligned}$$

where the first inequality follows from monotonicity of M and second follows from (24). Hence, $m(\beta(v')) > m(\beta(v))$. □

□

Proof of Proposition 6. To simplify the exposition, we assume that $r = v_L$[17]. Let, as before, v_1 denotes a valuation such that $v_1 - c(v_1) = v_L$ ($v_1 = v_H$ if $v_H - c(v_H) > v_L$), and let $\tilde{v}$ be an arbitrary valuation such that $\tilde{v} \in (v_L, v_1)$. Then define a density

[17] The proof is easily extended to the case of $r > v_L$.

function f as

$$f(v) = \begin{cases} a \text{ if } v \in [v_L, \tilde{v} - \epsilon] \\ a + \frac{(k-1)a}{\epsilon}(v - \tilde{v} + \epsilon) + ka \text{ if } v \in (\tilde{v} - \epsilon, \tilde{v}) \\ ka \text{ if } v \in [\tilde{v}, v_H] \end{cases}$$

where $k > 1$, $\epsilon \in (0, \tilde{v} - v_L)$ and $a = 1/(\epsilon(k-1)/2 + k(v_H - v_L) + (1-k)(\tilde{v} - v_L))$ to guarantee that the density function f integrates to 1. Given the specific form of a, we may always choose ϵ small enough such that $n\epsilon + \frac{1}{ka} < v_H - v_L$ (which is equivalent to $\epsilon < 2(k-1)(\tilde{v} - v_L)/(2nk + k - 1)$).

Note that $F(v) = a(v - v_L)$ for all $v \in [v_L, \tilde{v} - \epsilon]$ and $F(v) = 1 + ka(v - v_H)$ for all $v \in [\tilde{v}, v_H]$. Therefore,

$$\beta_{SPA}(v) = v + \frac{F(v)}{(n-1)f(v)} = \begin{cases} \frac{1}{n-1}(nv - v_L) \text{ , if } v \in [v_L, \tilde{v} - \epsilon] \\ \frac{1}{n-1}(nv - v_H + \frac{1}{ka}) \text{ , if } v \in [\tilde{v}, v_1] \end{cases}$$

Since $n\epsilon + \frac{1}{ka} < v_H - v_L$, $\beta_{SPA}(\tilde{v} - \epsilon) > \beta_{SPA}(\tilde{v})$ and hence, $\beta_{SPA}(v)$ is non-monotonic.

□

Proof of Proposition 7. Under the optimal taxation the expected utility of type v buyer should satisfy:

$$u^*(v) = \max_{v'} F^{n-1}(v')((1-t)v - \beta^*_{FPA}(v'))$$

Taking into account that maximum is achieved at $v' \leq (1-t)v$ and applying the Envelope Theorem yields:

$$u^{*\prime}(v) = (1-t)F^{n-1}(v') \leq (1-t)F^{n-1}((1-t)v)$$

Under the flat taxation, the net of taxes type v buyer's valuation equals to $(1-t)v$. Then in the first-price auction with symmetric equilibrium β_{FPA} the expected utility of type v buyer equals to

$$u(v) = \max_{v'} F^{n-1}(v')((1-t)v - \beta_{FPA}(v))$$

Since maximum is achieved at $v' = v$, applying the Envelope Theorem results in:

$$u'(v) = (1-t)F^{n-1}(v)$$

Since in both cases final allocation is efficient, the difference between seller's expected payoff under optimal and flat taxation equals to:

$$\begin{aligned} n\int_{v_L}^{v_H} u(v) - u^*(v)dF(v) &= n\int_{v_L}^{v_H} (u^*(v) - u(v))d(1-F(v)) \\ &= n\int_{v_L}^{v_H} (u'(v) - u^{*\prime}(v))(1-F(v))dv \\ &= n(1-t)\int_{v_L}^{v_H} (F^{n-1}(v) - F^{n-1}((1-t)v))dv \end{aligned}$$

where the second inequality follows from integration by parts. □

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