SEQUENTIAL SECOND PRICE AUCTIONS WITH BUDGET CONSTRAINED BIDDERS

GAGAN GHOSH & HENG LIU

Abstract. We study an auction game in which two units of a good are sold via two second price auctions sequentially. Bidders value the units identically and have one of two budget levels, high or low. Bidders do not know each others budgets. We show that this game has a unique symmetric equilibrium in which the probabilistic presence of high budget bidders can make bidders bid more aggressively in the first auction, thus lowering prices in the second. As a result if the possibility of competition from high budget bidders is large, then the equilibrium strategies generate declining prices.

1. Introduction

In many real-world auctions, such as those for wines, condominiums and lots of fish, multiple units of an object are sold sequentially in rapid succession. Bidders who are interested in buying more than one unit need to bid in each auction separately. While these sequential auctions are single object auctions by themselves, there are obvious links that connect them. One of these may be financial constraints faced by bidders, since the more a bidder spends in one auction the less she has available to spend in all following sales. In the presence of such links, the standard results from single-unit auction theory cannot be applied to analyze or predict behavior of bidders.

Budget constraints may exist in auctions for reasons such as incomplete capital markets or strategic considerations.[1] We take their presence as given and analyze how budgets affect

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[1]For example, see Börgers and Dustmann (2005) and Bulow et al. (2009).

the equilibria in sequential auctions. The introduction of budgets in a sequential auction adds a new strategic dimension to the game. On the one hand, bidders realize that if they attempt to win early auctions by bidding more, they reduce their chances of winning later, since they deplete their budgets by the price of the unit in the first auction. On the other hand, bidders might also be interested in bidding up the price in early auctions in an attempt to deplete the budgets of their rivals so that they can win later rounds at lower prices. This paper explores these strategic considerations in the presence of budgets.

Specifically, we study the sale of two identical items through two second price auctions (SPA) sequentially. Bidders are assumed to have an identical common value for winning each item. The bidders' ability to spend in each auction is constrained by exogenously given budget levels. These budget levels can be different across bidders. Bidders know their own budget levels but only have partial information about the budgets of others. Our main contribution is the discovery of the unique symmetric Bayes Nash equilibrium in the sequential auctions with private budgets (Propositions 3.1, 5.7 and 5.10).

For the case of three bidders (which can be generalized to the case with more than three bidders, as shown in section 5.2), the equilibrium takes on different forms depending on the parameter values. A key feature of the equilibrium is that the probabilistic presence of high budget bidders makes the other bidders bid aggressively, especially in the first round. As the likelihood of facing high budget bidders increases, bidders bid even more aggressively (section 4.2). The reason for this aggressive behavior is that since there are three bidders, there is a positive probability that one of the losers from the first auction will be a high budget bidder. As such, this bidder will hold an advantage in the second round. This makes the bidders bid more in the first round. In some cases, this aggressive bidding leads to a higher price in the first auction than that in the second. It in turn provides a possible explanation of the declining price anomaly in sequential auctions of identical items, which was first noted by Ashenfelter (1989) and later substantiated in other papers.[2]

[2]see McAfee and Vincent (1993), Beggs and Graddy (1997), Van den Berg et al. (2001), and Liu (2011) among others.

The literature has focused on two possible sets of explanations for the declining price anomaly. The first is based on some degree of heterogeneity between the objects for sale.[3] This explanation does not quite apply in a natural way to auctions of identical items such as wines, or almost identical items like flowers. The second set of explanations is based on auction specific institutional heterogeneity. For example Ginsburgh (1998) suggests that in wine auctions bidders are not the final consumers of the good and hence may be using sub-optimal strategies. Menezes and Monteiro (1997) show the prices may decline between auction rounds if the bidders have to pay a participation fee and do not know the number of bidders in the auction. The issue with these institutional explanations is that they only apply in specific cases. For example, while there might be some absentee bidders in wine auctions, it does not explain declining prices in the Dutch rose auctions (Van den Berg et al. (2001)). In contrast to these papers, our paper suggests an intuitive channel through which prices may decline in equilibrium, in general.

More recently, Mezzetti (2011) shows a different kind of risk aversion, called aversion to price risk, can explain declining prices in sequential auctions. Bidders prefer paying a certain price rather than buying a good with the same expected value for a variable price. The author also says that such aversion to price risk might be driven by budget constraints (page 991). However a model with budgets was not explicitly studied. As such, we think it is imperative to study sequential auctions with budgets. In a recent working paper Rosato (2015) shows that expectations-based reference-dependent preferences and loss aversion can explain the declining price anomaly.

From a theoretical perspective, our paper relates to a small set of papers on multi-unit sequential auctions with budget constrained bidders. Pitchik and Schotter (1988) study an auction environment with complete information where two bidders are interested in buying two objects. The authors then use a lab experiment to validate the predictions of the model. Both in the theoretical and the experimental study, the authors were interested in

[3]see Bernhardt and Scoones (1994),Engelbrecht-Wiggans (1994), Gale and Hausch (1994) and Kittsteiner et al. (2004).

the sequence of sales. Similar to this paper Benoît and Krishna (2001) study a complete information auction game in which a seller is trying to sell the objects to bidders who are budget constrained. The main emphasis of this paper was the sequence of sale and its effect on revenue. They were able to extend the model of Pitchik and Schotter (1988) in several directions. In our case, we have an incomplete information environment and the objects are identical. Hence the sequence doesn't matter and as such our paper differs considerably from the above.

Perhaps the closest paper to ours, in terms of scope, is Pitchik (2009). She studies a case in which two budget constrained bidders with private values are competing for two objects in sequential auctions. Both the budgets and valuations are private information, which is modeled using a single dimensional parameter. The main focus of her paper is the implication of budgets on the order sales, which does not matter in our paper since we have identical values.

There are some crucial differences between the current paper and Pitchik (2009). The most important one is that in Pitchik (2009) the model has two bidders, whereas we solve an auction model for three bidders and show that the equilibrium is qualitatively different form the two bidder version in section 5.1.[4] In Pitchik (2009), after the first auction is over the loser knows the budget of the winner (page 931). With three or more bidders, such an assumption doesn't make the problem any more tractable as there is always another bidder who did not win the first auction. A result in Pitchik (2009) is that when bidders' value the items identically, and the bidding function is ordinarily equivalent to the income function, then the expected price of good 2 is higher than that of good 1 (Corollary 5).[5] In contrast to this result, we show in section 5.1 that the law of one price holds. The reason for this divergence in results is that in our model when there are only two bidders the bids in the first round are not increasing in budgets. In Pitchik (2009), bids have to be increasing in

[4] Other differences include, how incomplete information is modeled, and the fact that Pitchik (2009) has multiplicity of equilibria, where as we have uniqueness.

[5] $f(\cdot)$ and $g(\cdot)$ are ordinally equivalent to each other if whenever $f(x') > f(x'')$ implies $g(x') > g(x'')$.

budget levels in order to generate increasing prices. In contrast to the two bidder model, we show that expected prices can decline in the model with three bidders or more, which underscores the importance studying a model where the number of bidders is larger than the number of objects.

Papers such as Borgs et al. (2005), Dobzinski et al. (2012) and Hafalir et al. (2012) take a mechanism design approach to studying multi-unit auctions with budgets. Such an approach is also popular among computer scientists studying these problems. This line of research seems to underscore the need to understand how budgets affect bidding behavior in auctions.

The rest of the paper is organized as follows. In section 2 we describe the model, assumptions made in it as well as the set of strategies in which we will look for an equilibrium. The main result of the paper is stated in proposition 3.1. The rest of section 3 is devoted to proving the existence and uniqueness of the equilibrium. In section 4 we carry out some comparative static exercises and show precisely how prices can decline in equilibrium in these auctions. Finally in section 5 we study a few extensions of the model. In sections 5.1 and 5.2 we extend the model to the cases of two and more than three bidders respectively. In the former, we show that there is a qualitative difference between the equilibrium when we go from two to three bidders. However for the case of more than three bidders, our conjecture is that our results for the the three bidder case go through. In section 5.3 we show that our results are robust to changes in parameter specifications.

2. Model and Strategies

Three bidders compete for two units of a good. The units are sold sequentially in second price auctions (SPA). Each bidder values each unit at v. The bidders also have one of two budget levels, w_h, w_l, with $w_h > w_l$. A bidder has the high budget level with probability ρ. Also, we assume $v > w_h$. We make this assumption to focus our attention on effect of budget constraints in these auctions. However, this assumption does not imply that the valuation of the good has no effect on strategies, as can be seen from proposition 3.1. In each auction the bidder with the highest bid wins the auction and ties are broken via a fair coin-flip.

Let i represent a bidder, $j \in \{l, h\}$ a bidder's type in terms of initial budget level and k the auction number. Let m^i_{jk} be bidder i, type j's budget in auction k and b^i_{jk} be bidder i, type j's bid in auction k. Finally suppose p_k is the price of the unit in round k. Then

$$m^i_{jk} = \begin{cases} w_j; \text{ if } k = 1 \\ w_j; \text{ if } k = 2 \text{ and } i \text{ lost round 1} \\ w_j - p_1; \text{ if } k = 2 \text{ and } i \text{ won round 1} \end{cases}$$

It must be the case that $b^i_{jk} \in [0, m^i_{jk}]$ for all i, j and k. Let $\mathcal{B}^i_{jk}$ be the Borel sigma algebra on $[0, m^i_{jk}]$. A behavioral strategy for a player i in each round is a probability transition function $\sigma^i_{jk} : \mathcal{B}^i_{jk} \to [0, 1]$. Such a strategy generates bidder specific cumulative distributions of bids given by $F^i_{jk}(\cdot) = \sigma^i_{jk}([0, b])$. We will characterize the equilibrium using these bid distributions.

3. Equilibrium Characterization

Since the game we are considering is a finite sequential game, it is easiest to begin with the final stage of the game, which is the second auction. Using the equilibrium behavior of the bidders in the second auction, which is done in the next sub-section, we will work backwards to construct an equilibrium of the first auction. In this section and the next we assume $w_h < 2w_l$. In equilibrium, this inequality ensures that a bidder can only win one auction, which allows us to obtain a 'clean' closed form of the equilibrium. The qualitative features of an equilibrium where $2w_l < w_h$ remain identical and we discuss this case in section 5.3.[6]

3.1. Second Auction.

The final auction is essentially a single unit SPA with budget constrained bidders. It is well known that it is weakly dominant to bid one's valuation in a

[6]When $w_h < 2w_l$ a bidder wins at most one auction. This implies that a model where bidders have single-unit demands can be nested in our model.

static SPA. With budget constraints, this result is altered slightly, as show in Che and Gale (1998). In the second auction bidders will bid b^i_{j2}, given by

$$b^i_{j2} = \min\left\{m^i_{j2}, v\right\} = m^i_{j2},$$

where the second equality follows from $w_h < v$. Note that, the above is equivalent to

$$F^i_{j2}(b) = \begin{cases} 1; \text{ if } b \geq m^i_{j2} \\ 0; \text{ otherwise} \end{cases}$$

3.2. **First Auction.** The bid distribution in the second auction is degenerate. Therefore, from now on we will use $F^i_j(\cdot)$ to represent the bid distribution used by bidder i of type j in the first auction. We will also restrict attention to symmetric equilibria, thereby suppressing the superscript. (All results should be extendable to asymmetric distributions). Therefore $F_j(\cdot)$ represents the first round bid distribution of bidder type j. We state our main result in the following proposition.

Proposition 3.1. *In the unique symmetric Nash equilibrium of the game the low budget bidders always bid their budget in the first round. That is*

$$F_l(b) = \begin{cases} 1; \text{ if } b \geq w_l \\ 0; \text{ otherwise} \end{cases}$$

For the high budget bidders, the equilibrium bid distributions will take one of three forms.[7] *Define $G(b), \underline{w}$ and $\bar{w}$ as follows*

$$G(b) = \frac{2(1-\rho)(b-w_l)}{\rho(v+w_h-2b)};\ \underline{w} = \frac{6(1-\rho)}{6-5\rho}w_l + \frac{\rho}{6-5\rho}v;\ \bar{w} = \frac{2(1-\rho)}{2-\rho}w_l + \frac{\rho}{2-\rho}v$$

(i) if $w_h \leq \underline{w}$, then

$$F_h(b) = \begin{cases} 1\text{: if } b \geq w_h \\ 0; \text{ otherwise} \end{cases}$$

[7]See Figures 1, 2 and 3 for illustrations of the bid distributions

(ii) If $\underline{w} < w_h < \bar{w}$, then

$$F_h(b) = \begin{cases} G(b); \text{ if } b \in (w_l, \hat{b}] \\ G(\hat{b}); \text{ if } b \in (\hat{b}, w_h) \\ 1; \text{ if } b \geq w_h \\ 0; \text{ otherwise} \end{cases}$$

where

$$\hat{b} = \frac{2(1-\rho)w_l + \rho(1-\Delta_h)(v+w_h)}{2\left((1-\rho)+\rho(1-\Delta_h)\right)}; \; \Delta_h = \frac{3}{2}\left(1 - \frac{2(1-\rho)(w_h-w_l)}{\rho(v-w_h)}\right)$$

(iii) If $w_h \geq \bar{w}$, then

$$F_h(b) = \begin{cases} G(b); \text{ if } b \in (w_l, \bar{b}] \\ 1; \text{ if } b \geq \bar{b} \\ 0; \text{ otherwise} \end{cases}$$

where

$$\bar{b} = (1-\rho)w_l + \frac{\rho(v+w_h)}{2}$$

In the second auction, all bidders bid their remaining budgets.

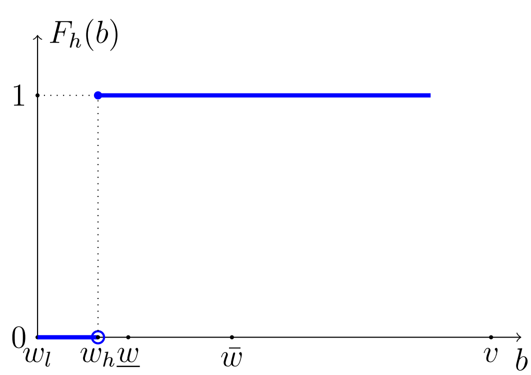

FIGURE 1. Case (i): $w_h \leq \frac{6(1-\rho)}{6-5\rho} w_l + \frac{\rho}{6-5\rho} v$

We prove this proposition through a series of lemmas. All the proofs can be found in appendix A. First, we show that if an equilibrium exists then it must take the form shown in the proposition (lemmas 3.2-3.10). To prove existence we show that the strategies stated

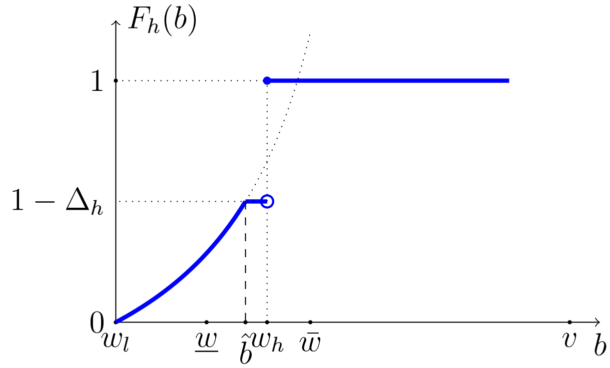

FIGURE 2. Case (ii): $\frac{6(1-\rho)}{6-5\rho}w_l + \frac{\rho}{6-5\rho}v < w_h < \frac{2(1-\rho)w_l}{2-\rho} + \frac{\rho}{2-\rho}v$

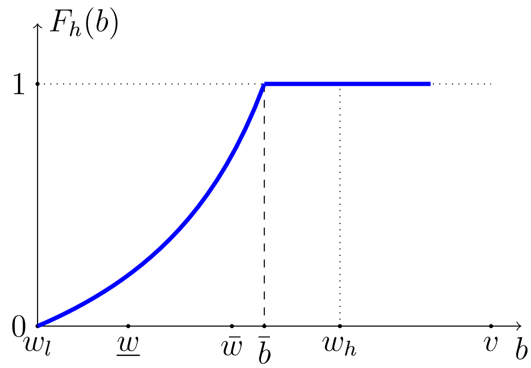

FIGURE 3. Case (iii): $w_h \geq \frac{2(1-\rho)}{2-\rho}w_l + \frac{\rho}{2-\rho}v$

in the proposition are in fact an equilibrium (lemma 3.11). Let $F_l(\cdot)$ and $F_h(\cdot)$ be a pair symmetric equilibrium bid distributions. Let $\Pi_i(b)$ be the total payoff (sum of expected payoffs across two auctions) to a bidder type i from bidding bid b in the first auction if the other bidders are bidding according to some F_h, F_l. Let $\Pi_i^{sq}(b)$ be a bidder i type's expected payoff from competing against bidder types $s, q \in \{l, h\}$. Then bidder i type's expected payoff from bidding b in the first auction is given by

$$\Pi_i(b) = \Pi_i^{ll}(b) + \Pi_i^{lh}(b) + \Pi_i^{hh}(b) \tag{1}$$

Depending on who the bidder is competing against the arguments of the payoff function will change. A crucial point to note is that the payoff function's properties are inexorably linked to the properties of the bid distributions. We begin by first proving a property about the continuity of the bid distributions.

Lemma 3.2. *There does not exist any $b \neq w_l, w_h$ such that $\lim_{\varepsilon \to 0} F_i(b) - F_i(b-\varepsilon) > 0$ for any i. That is the equilibrium bid distributions can not have any atoms in it other than at the budget levels.*

Let $E_i \subset [0, w_i]$ be the set of bids over which the bidder randomizes in equilibrium in the first auction. For sure, if $b, b' \in E_i$, then $\Pi_i(b) = \Pi_i(b') \geq \Pi_i(b'')$, for all b''. Now, define a set $\operatorname{supp} F_i$ as follows.

Definition 3.3. $\operatorname{supp} F_i = \{b \mid F_i(b+\varepsilon) - F_i(b-\varepsilon) > 0, \text{ for all } \varepsilon > 0\}$.

Notice that $\operatorname{supp} F_i$ is a closed set, since it is the closure of the set of bids that a bidder type i randomizes over in the first auction. That is $\operatorname{supp} F_i = \bar{E}_i$. The next lemma makes an observation about $\operatorname{supp} F_i$.

Lemma 3.4. *If $b \in \operatorname{supp} F_i$, and there is no atom at b, then $\Pi_i(b) \geq \Pi_i(b')$, for all b'.*

Before we state the remaining results, a note about the payoff function. As long as two or more bidders bid more than zero in the first round, in all cases but one, a bidder can only win one auction. The only case where the same bidder might win both auctions, is when there are two low budget bidders and one higher budget bidder. However, for now, we assume that even in this case, a bidder can only win one auction. We will show that this is indeed the case in the unique Nash equilibrium of the game, under our parametric restrictions, i.e. $w_h < 2w_l$. As mentioned previously, the other case is analyzed in section 5.3.

The next lemma shows that in equilibrium the supports of the bid distributions cannot overlap. This allows us to rule out many types of equilibria.

Lemma 3.5. *There does not exist any $b < w_l$ such that $b \in \operatorname{supp} F_i$ for both $i = l, h$.*

This lemma rules out many types of equilibria. Essentially, in equillirbrium the set of bids over which the bidders randomize, E_i, are non-intersecting, other than possibly at w_l. We eventually prove that $E_l \cap E_h = \emptyset$.

Lemma 3.6. *There does not exist $b < w_l$ such that $F_l(b) > 0$. That is the low budget bidders always bid their budget in the first round.*

If an equilibrium exists, then Lemma 3.6 proves the first part of proposition 3.1. Next we will show that if an equilibrium exists then the strategies of the high budget bidder must take the form stated in the proposition. We begin by showing that the high budget bidders bid above w_l in equilibrium.

Lemma 3.7. *There does not exist $b < w_l$, such that $b \in \operatorname{supp} F_h$. Also, the high budget bidder will never bid w_l in the first auction.*

At this point we prove the conditions for the existence of a degenerate equilibrium where the low budget bidders bid their budgets, which must happen in any equilibrium, and the high budget bidders also bid their budgets.

Lemma 3.8. *In equilibrium $F_h(b) > 0$, where $b < w_h$ if and only if $w_h > \underline{w}$*

Lemma 3.8 proves part (i) of proposition 3.1. The lemma also shows that if $w_h > \frac{6(1-\rho)}{6-5\rho} w_l + \frac{\rho}{6-5\rho} v$, the degenerate equilibrium can not exist. Therefore, in equilibrium, the high budget bidder must submit bids less that w_h in the first auction with positive probability. From Lemma 3.2 we also know that the bid distribution has be continuous everywhere except at w_h. This implies that if $w_h > \frac{6(1-\rho)}{6-5\rho} w_l + \frac{\rho}{6-5\rho} v$, the optimal strategy of the high budget bidder will include mixing continuously over some interval. She may still choose to submit w_h with positive probability, but that will depend on parameter values.

Lemma 3.9. *If $w_h \geq \bar{w}$ and an equilibrium exists, then the high budget bidder will play a mixed strategy in equilibrium in which she randomizes over an interval. The strategy is given by case (iii) in proposition 3.1*

In view of Lemmas 3.8 and 3.9, the only case left to consider is where $\frac{6(1-\rho)}{6-5\rho} w_l + \frac{\rho}{6-5\rho} v < w_h < \frac{2(1-\rho) w_l}{2-\rho} + \frac{\rho}{2-\rho} v$. The next lemma establishes the form the equilibrium must take, if it exists, under this case.

Lemma 3.10. *If $\underline{w} < w_h < \bar{w}$ and an equilibrium exists, then the high budget bidder bids her budget level with a psoitive probabilty but also mixes over an interval of bids. The strategy is given by case (ii) in proposition 3.1*

Lemmas 3.2-3.10 show that if an equilibrium exists, then it must take the form as stated in proposition 3.1. In order to complete the proof of the proposition we need to show that these strategies do indeed constitute an equilibrium. This we do in the next lemma, which serves as a proof of existence.

Lemma 3.11. *The strategies, in terms of the bid distributions $F_h(\cdot)$ and $F_l(\cdot)$ as stated in proposition 3.1 constitute an equilibrium.*

Let us provide some intuition for the structure of the equilibrium. First consider low budget bidders. In the second auction, conditional on winning, they have to pay of price of w_l. A low budget bidder can win the second round if and only if (i) she lost the first round and (ii) there is at most one high budget bidder. Now consider a low budget bidder's incentives to bid more in the first auction. Suppose she is bidding $b < w_l$. If she wins, her payoff is greater than $v - b > v - w_l$. If she bids more, she increases her likelihood of winning the first auction but reduces her chances of winning the second. Since the payoff if the second is lower than the second in the first, she would prefer to bid more in the first, as long as it is feasible. Therefore, she would bid w_l in the first auction.

Now consider high budget bidders. Suppose all bidders are bidding their budgets in each auction. Then a high budget bidder has to decide whether she wants to lose the first auction (to a high budget bidder, thereby also 'getting rid of her') and try and win the second at (possibly) lower price of w_l. If w_h and w_l are close enough then a high budget prefers increasing her probability of winning, which she does by bidding as much as possible in the first auction. For higher levels of w_h, paying a price of w_l in the second auction becomes much more attractive and a high budget bidder may try to lose the first auction to other high budget bidders by putting positive probability on lower bids.

4. Comparative Statics and Price Paths

In this section, we examine how changes in budget levels and prior probabilities affect the high budget bidders' equilibrium bids and payoffs as well as the realized and expected price paths. Since in the second auction, all bidders bid their remaining budgets, we focus on the bids in the first auction.

4.1. Changes in w_h.

We know that the low budget bidders always bid their budgets. Hence the relevant comparative static to perform is with respect to the high budget level. The equilibrium behavior of the high budget bidders depends on the level of the high budget. Therefore any comparative static with respect to w_h does need to account for the different cases. We begin with a discussion on expected bids.

A bidding aspect economists are often interested in is how 'aggressive' bidding is in an auction. In order to answer this question we first need to define aggression. The conventional definition of aggressive behavior in auctions is when a bidder bids more. We call this *absolute aggression.* However, in an auction where bidders are budget constrained, if a bidder bids more than another, that doesn't necessarily imply that the other bidder is less aggressive, since the other bidder might be constrained by her budget. As such, we argue that that there is another relevant interpretation of aggression in our context. A bidder can be thought of as bidding relatively more aggressively than another if she bids a larger portion of her budget in the auction. We call this behavior *relative aggression.* Under this definition, the low budget bidders are relatively more aggressive than high budget bidders if $w_h > \underline{w}$. If $w_h \leq \underline{w}$, then both the bidder types are equally relatively aggressive. The high budget bidders are always more aggressive than the low budget bidders in absolute terms. In our opinion, both type of aggression need to be measured.

4.1.1. *Absolute aggression, stochastic dominance and first round bids.*

First, let us say something about absolute aggression. We already know that the high budget bidders are more aggressive than the low budget bidders in the first auction. We are interested in the behavior of high budget bidders as w_h changes. Since the form of the equilibrium depends on the value of w_h, we take the three cases separately.

Case (i) $w_h \leq \underline{w}$: In this case, all else being equal, the high type bidders bid more aggressively in the first auction with an increase in the the budget level in the sense of first order stochastic dominance. See Figure 4. Since the equilibrium bid distribution in this case is degenerate, we know that the expected bid of high budget bidder in the first auction is simply w_h.

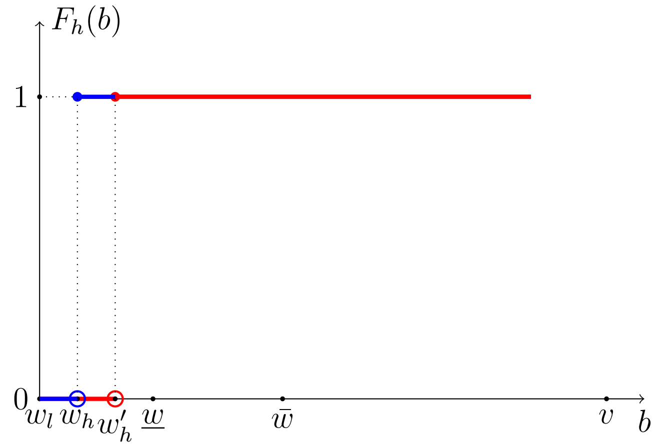

FIGURE 4. Case (i): $w_h < w'_h \leq \underline{w}$

Case (ii) $\underline{w} < w_h < \bar{w}$: When the budget level of the high type bidders, w_h is between $\underline{w}$ and $\bar{w}$, the bid distribution has an atom at w_h; while the support of the bid distribution expands with an increase in w_h, the probability mass at w_h decreases (See Figure 5). As such the bid distribution of the bidder with the higher budget does not dominate the distribution of the bidder with the lower budget level. Hence, we can not make a statement about which bidder is more aggressive using only the distribution.

In order to circumvent this issue, we calculate the expected bid in the first auction. If this expected bid increases with w_h then we can say that higher budget levels lead to more aggressive behavior in the first auction. Let Δ_h be the probability the bidder places on

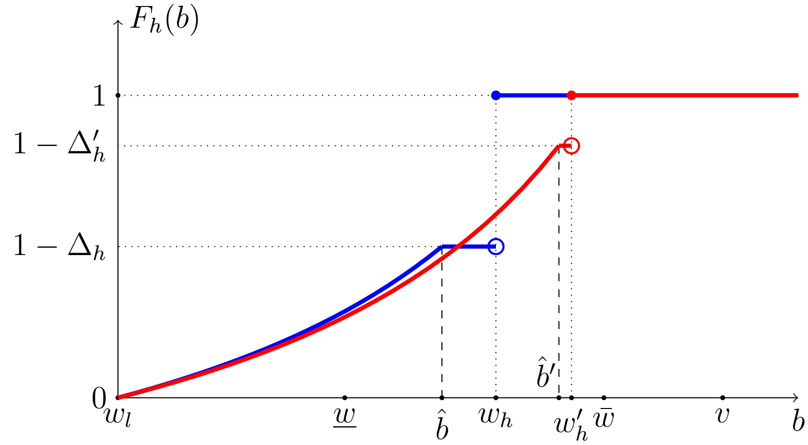

FIGURE 5. Case (ii): $\underline{w} \leq w_h < w'_h \leq \bar{w}$

bidding w_h. Then expected bid in the first round is given by

$$\mathbb{E}(b|w_h) = \int_{w_l}^{\hat{b}} x dF_h(x) + \Delta_h w_h, \tag{2}$$

where $\hat{b}$, Δ_h and $F_h(\cdot)$ are given in the statement of proposition 3.1. In the appendix we calculate this expectation and show it to be equal to the following:

$$\mathbb{E}(b|w_h) = \left(\frac{1-\rho}{2\rho}\right)(v + w_h - 2w_l)\ln\left(\frac{1-\rho}{1-\rho+\rho(1-\Delta_h)}\right) + \frac{v}{2}(1-\Delta_h) + \frac{w_h}{2}(1+\Delta_h). \tag{3}$$

We would like to know how $\mathbb{E}(b|w_h)$ changes with respect to w_h. However the derivative of the former with respect to the latter does not give us a clear answer. To see this note that as w_h increases, Δ_h decreases, which makes it hard to ascertain the sign of the derivative of $\mathbb{E}(b|w_h)$. Hence we resort to the use of a numerical exercise.

Example 4.1. Consider $v \in \{8, 10, 12\}$, $w_l = 4$ and $\rho \in \{0.25, 0.5, 0.75\}$. Figure 6 plots the $\mathbb{E}(b|w_h)$ with respect to various budget levels $w_h \in [\underline{w}, \bar{w}]$. The bounds of this support are calculated for each combination of v and ρ. We see that the expected bid in the first auction rises with w_h for any combination of v and ρ.

Case (iii) $w_h \geq \bar{w}$: This case is similar to case (i), except that the bidders are mixing continuously over an interval. As w_h increases, the equilibrium bid distribution of bidder

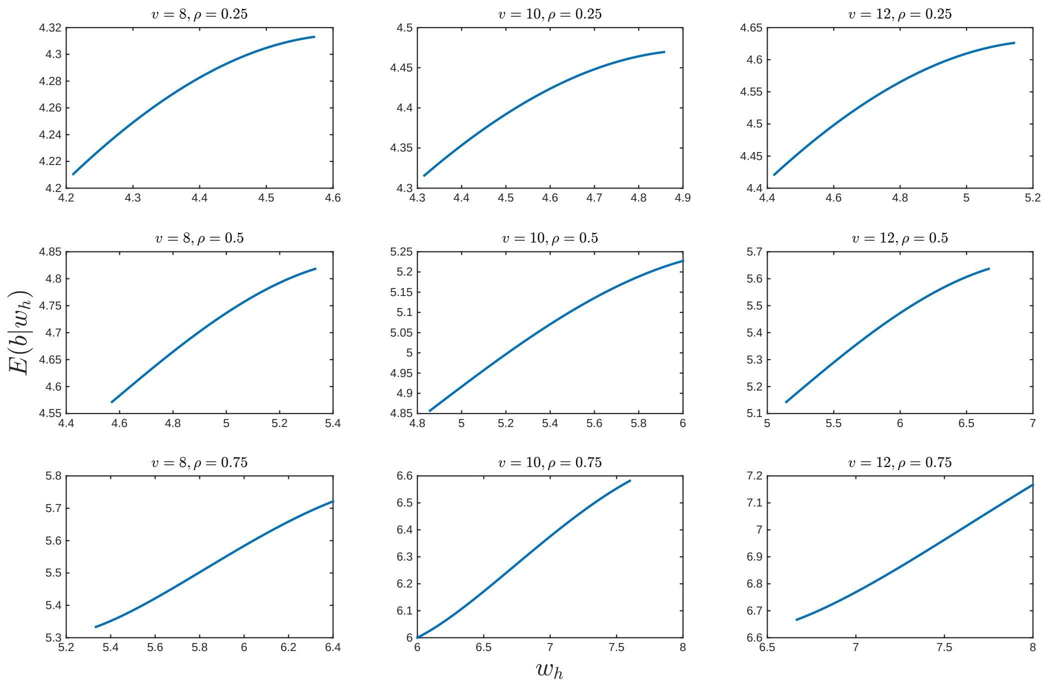

FIGURE 6. Expected bid in the first auction

with a higher budget level stochastically dominates one with a lower budget level in a first order sense. See Figure 7 for an illustration. The implication of this stochastic dominance is pretty straightforward- as w_h increases, the expected first round bid of the high budget bidders, $\mathbb{E}(b|w_h)$ also increases.

Using an equation almost identical to equation (2) we can in fact find the closed form for the expected bid. The only difference will be that in this case $\Delta_h = 0$, as the bidder bids her budget level with zero probability. Then we have

$$\mathbb{E}(b|w_h) = \int_{w_l}^{\hat{b}} x dF_h(x). \tag{4}$$

Using similar methods used in calculating $\mathbb{E}(b|w_h)$ in case (ii) we can calculate the value here as well. It turns out that in this case $\mathbb{E}(b|w_h)$ has a closed form solution which is linear in

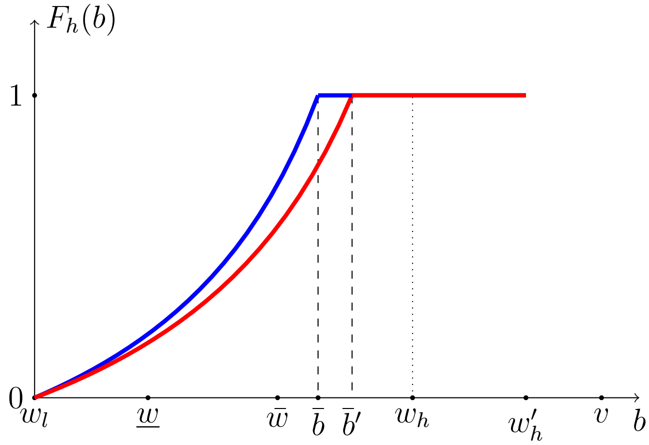

FIGURE 7. Case (iii): $w_h' > w_h \geq \bar{w}$

w_h. That is,

$$\mathbb{E}(b|w_h) = \left(\frac{1-\rho}{2\rho}\right)(v+w_h-2w_l)\left(\ln\left(\frac{v+w_h-2\bar{b}}{v+w_h-2w_l}\right)+\frac{(v+w_h)(2\bar{b}-2w_l)}{(v+w_h-2\bar{b})(v+w_h-2w_l)}\right)$$

Substituting the value of $\bar{b}$ from proposition 3.1 in the above equation and simplifying, we can see that $\mathbb{E}(b|w_h)$ is increasing in w_h.[8]

$$\mathbb{E}(b|w_h) = \left(\frac{1-\rho}{2\rho}\right)(v-w_l)\ln(1-\rho)+\frac{v}{2}+\left(\frac{(1-\rho)\ln(1-\rho)}{2\rho}+\frac{1}{2}\right)w_h \tag{5}$$

4.1.2. *Relative aggression.* To measure relative aggression, as we have defined it, we divide the expected bid submitted by a high budget bidder in the first round by her budget. Let $\alpha(w_h) = \frac{\mathbb{E}(b|w_h)}{w_h}$ be this ratio. In case (i) when $w_h \leq \underline{w}$, $\alpha(w_h) = 1$, since the high budget bidder always bids her budget. In case (iii), $\alpha(w_h)$ is strictly decreasing in w_h. This is obvious from equation (5).

For case (ii), it is not immediately clear whether $\alpha(w_h)$ is increasing or decreasing in w_h. Therefore, we again take the help of a numerical exercise.

Example 4.2. Using the same values as in Example 4.1, Figure 8 shows the levels of relative aggression for the high type bidders as measured by $\alpha(\cdot)$. For completeness, we show the

[8] $\frac{(1-\rho)\ln(1-\rho)}{\rho}+1>0$. If not then $-\ln(1-\rho) \geq \frac{\rho}{1-\rho}$. Both functions are equal to to zero at $\rho=0$ and the slope of the right hand side is always greater than the left which provides the contradiction

levels of $\alpha(\cdot)$ for all the cases, (i) $w_h \leq \underline{w}$, (ii) $\underline{w} < w_h < \bar{w}$ and (iii) $\bar{w} \leq w_h$. From the figures it is clear that $\alpha(\cdot)$ is strictly decreasing in cases (ii) and (iii).

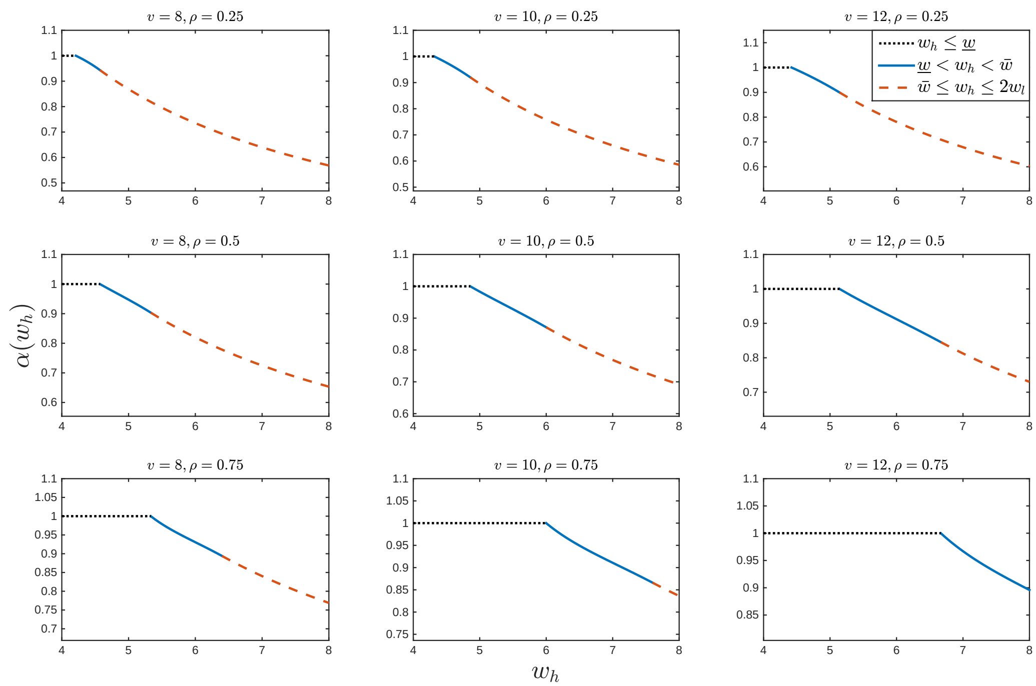

FIGURE 8. Relative aggression as a function of w_h

The inverse relationship between relative aggression and w_h is an interesting aspect of the equilibrium. The intuition for this is similar to the intuition behind the structure of the equilibrium. In the first auction, high budget bidders encounter a trade-off between winning the second auction at a low price (w_l), when there is at least one low budget opponent, against losing both auctions when both opponents have high budget. In other words, bidding more in the first auction leads to a higher probability of winning against bidders who are also high type but reduces the probability of winning the second auction against a low budget bidder (since winning the first auction at a price higher than w_l, which will happen equilibrium, means she cant win the second auction). For fixed w_l and v as w_h increases, $v - w_l$ as a ratio of $v - w_h$ is increasing. That is, winning against a low budget bidder in the second auction is becoming more attractive. This compels a high budget bidder to 'back-off' in his first round bid. By backing off a high budget bidder increases her chances of competing against a low budget bidder in the second auction.

4.2. **Changes in ρ.** Recall that the trade-off for a high budget bidder in the first auction is between (possibly) winning the second auction at a low price (w_l), and (possibly) losing both auctions when both opponents have high budget. As such, different probabilities of facing high budget bidders affect the equilibrium behavior of a high budget bidder. Intuitively, as ρ increases the probability of losing both auctions starts outweighing the expected gain from being able to compete against a low budget bidder in the second auction.

Consider changes in the prior probability ρ. We observe that as ρ increases, both $\underline{w}$ and $\bar{w}$ increase.[9] As $\underline{w}$ increases, the range of w_h over which the high budget bidder bids her entire budget, $(w_l, \underline{w})$, increases, as can be seen in Figure 9. Intuitively, a higher ρ means that there is more competition from high budget bidders. As a result for a larger range of high budget levels, a high budget bidder will bid as aggressively as she can, i.e. bid her budget. In contrast, a lower ρ means less competition from high budget bidders, and a high budget bidder could bid below her budget (still above w_l) and increase her chances of competing against a low budget bidder in the second auction.

Another related observation is that as ρ increases, the high budget bidder becomes relatively more aggressive. We can see this from Figure 10, in which $v = 10$ and w_l and we vary ρ. The intuition for this is similar to that given in the previous paragraph. With increasing ρ high budget bidders, need to be more aggressive as they are more likely to be competing against other high budget bidders.

4.3. **Price Path.** Finally, we investigate the price paths across auctions. Note that regardless of the parameter values, if there are at least two low budget bidders among the three bidders, then the realized prices for the two auctions are constant, which are equal to w_l; if

[9]Note that $\frac{d\underline{w}}{d\rho} = \frac{6}{(6-5\rho)^2}(v - w_l) > 0$ and $\frac{d\bar{w}}{d\rho} = \frac{2}{(2-\rho)^2}(v - w_l) > 0$.

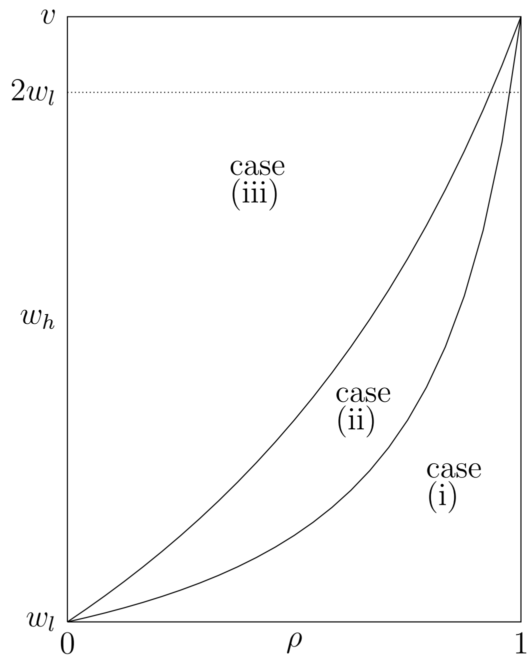

FIGURE 9. Parameters: ρ and w_h

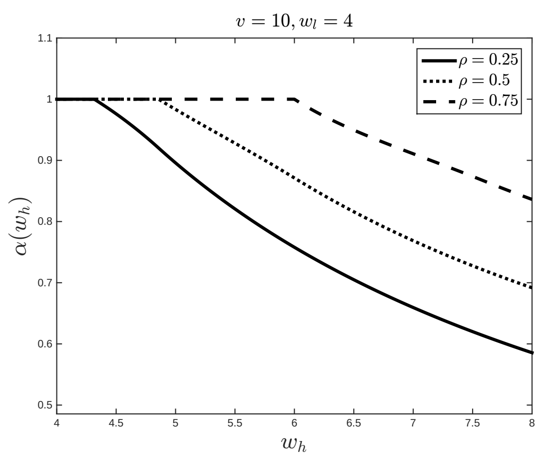

FIGURE 10. Relative aggression as ρ changes

there is only one low budget bidder, then we obtain a decreasing price path where the price of the second auction is w_l; if all bidders have high budget, then the equilibrium features an non-decreasing price path, with the price of the second auction being w_h. Moreover, in case (i) of the equilibrium where high budget bidders bid the entire budget, the prices either stay constant or decrease; in case (ii) of the equilibrium, since the equilibrium is continuous with respect to the parameters w_h and ρ, we have a decreasing expected price path if the probability that high budget bidders bid their entire budget is large enough. This result on the decreasing price path is summarized in the following proposition.

Proposition 4.3. *For each w_h, the expected price in the first auction exceeds the price in the second auction if $\rho > \bar{\rho}(w_h)$ for some $\bar{\rho}(w_h) \in (0,1)$.*

Proof. This follows from Figure 9 and the discussion in the previous paragraph. Furthermore, by continuity of the equilibrium with respect to the parameters w_h and ρ, for each w_h, $\bar{\rho}(w_h)$ satisfies

$$w_h < \frac{2(1-\bar{\rho}(w_h))w_l}{2-\bar{\rho}(w_h)} + \frac{\bar{\rho}(w_h)}{2-\bar{\rho}(w_h)}v$$

□

Finally, the following numerical example shows that there can be a decreasing expected price path in case (iii) of the equilibrium. Let p_1 be the price of the first auction and p_2 the price of the second auction. For any $m, k \geq \mathbb{N}_+$ and $k \leq m$, denote $X_m^{(k)}$ the random variable that is the k-th highest of m i.i.d. draws from the distribution F_h. Then the expected difference between p_1 and p_2 is:

$$\begin{aligned}\mathbb{E}(p_1 - p_2) &= 3\rho^2(1-\rho)\left(\mathbb{E}X_2^{(2)} - w_l\right) + \rho^3\left(\mathbb{E}X_3^{(2)} - w_h\right)\\ &= \rho^2\left[3(1-\rho)\left(\mathbb{E}X_2^{(2)} - w_l\right) + \rho\left(\mathbb{E}X_3^{(2)} - w_h\right)\right]\end{aligned}$$

Example 4.4. Consider $v = 40$, $w_l = 15$, and $w_h = 20$. When $\rho \leq 1/3$, high budget bidders' bids follow the atomless distribution F_h as in case (iii) of the equilibrium. In the following table (Table 1), we compute the expected difference between the price of the first auction and the price of the second auction, for various values of $\rho \in (0, 1/3)$.

5. Extensions

5.1. **Two Bidders.** When there are two bidders, or more generally, the number of bidders equals the number of units, we show that there is still a unique symmetric equilibrium in

ρ	$\mathbb{E}(p_1 - p_2)$
0.05	0.00125
0.1	0.01
0.2	0.08
0.3	0.27
1/3	10/27

TABLE 1. Declining prices

the sequential auction, yet it is qualitatively different from the case in which there are more bidders than units. Specifically, the next proposition shows that both types of bidders bid half of the expected budget in the first auction. Furthermore, in this case the "law of one price" holds, that is, the expected prices of both auctions are the same.

Proposition 5.1. *Suppose that* $w_l > \frac{2-\rho}{3-\rho} w_h$*. There is a unique symmetric equilibrium, in which bidders submit the same bid* $b^1 = \rho \frac{w_h}{2} + (1-\rho)\frac{w_l}{2}$ *in the first round and bid* $b_i^2 = \min\{v, m_i\}$*, where* m_i *is their budget left over in the second round. Moreover, the expected price of the first auction is equal to the expected price of the second auction.*

Proof. Let Π be the payoff to a bidder from bidding b^1, b^2.

$$\Pi = (1-\rho)\left(\frac{1}{2}(v - b^1) + \frac{1}{2}(v - w_l + b^1)\right) + \rho\left(\frac{1}{2}(v - b^1) + \frac{1}{2}(v - w_h + b^1)\right)$$
$$= v - \left((1-\rho)\frac{w_l}{2} + \rho\frac{w_h}{2}\right).$$

Suppose the bidder bids $\varepsilon (> 0)$ more in the first auction and the other bidder plays according to prescribed strategies. Then she wins first auction for sure and loses the second auction for sure, since $w_l > \frac{2-\rho}{3-\rho} w_h$. Correspondingly, her payoff from this deviation is

$$\Pi_+ = v - b^1 = \Pi.$$

Now suppose the bidder bids ε less in the first auction. He will lose the first auction for sure but win the the second auction for sure, and his payoff from this deviation is

$$\begin{aligned}\Pi_- &= (1-\rho)\left(v - (w_l - (b^1-\varepsilon))\right) + \rho\left(v - (w_h - (b^1-\varepsilon))\right)\\ &= v - \left((1-\rho)\frac{w_l}{2} + \rho\frac{w_h}{2}\right) - \varepsilon < \Pi.\end{aligned}$$

Finally, to show that the "law of one price" holds, note that the price of the first auction, p_1, is

$$p_1 = \rho\frac{w_h}{2} + (1-\rho)\frac{w_l}{2},$$

and the expected price of the second auction is

$$\begin{aligned}&(1-\rho)^2(w_l - p_1) + 2\rho(1-\rho)\left[\frac{1}{2}(w_l-p_1) + \frac{1}{2}(w_h - p_1)\right] + \rho^2(w_h - p_1)\\ &= (1-\rho)w_l + \rho w_h - p_1 = \rho\frac{w_h}{2} + (1-\rho)\frac{w_l}{2} = p_1.\end{aligned}$$

□

As a remark, we note that the above equilibrium and proof go through, whenever the number of bidders equals the number of units, and the units are sold sequentially one in every period, i.e., in the unique symmetric equilibrium, all bidders who have not won a unit will submit the same bid, which equals half of the expected budget throughout.

5.2. **More Than Three Bidders.** In this section, we explain how to generalize the results to the case with more than three bidders.

Suppose that two units of an identical good are sold sequentially in second price auctions and there are $N = n+1$ bidders ($n \geq 2$). Each bidder values each unit at v. The bidders also have one of two budget levels, w_h or w_l, and a bidder has the high budget level with probability ρ. Following the same argument as in the previous sections, we will consider the equilibrium in which low budget bidders always bid their entire budget, and the bid distribution of high budget bidders has at most one atom at w_h.

Here we adapt the derivations of equilibrium in Section 3. Let $F_h^{(n)}(\cdot)$ be the high budget bidders' bid distribution. Then the high budget bidders' expected payoff, $\Pi_h(b)$, is given by

$$\Pi_h(b) = \binom{n}{0}(1-\rho)^n(v-w_l) + \binom{n}{1}\rho(1-\rho)^{n-1}\left[\int_{\bar{b}}^{b}(v-x)dF_h^{(n)}(x) + (1-F_h^{(n)}(b))(v-w_l)\right]$$
$$+\sum_{k=2}^{n}\binom{n}{k}\rho^k(1-\rho)^{n-k}\left[\int_{\bar{b}}^{b}(v-x)k(F_h^{(n)}(x))^{k-1}f_h(x)dx\right.$$
$$\left.+(1-(F_h^{(n)}(b))^k)\left(\frac{v-w_h}{k}\right)\right]$$

with the corresponding first-order condition:

$$\frac{d\Pi_h(b)}{db} = n\rho(1-\rho)^{n-1}f_h^{(n)}(b)(w_l-b)$$
$$+\sum_{k=2}^{n}\binom{n}{k}\rho^k(1-\rho)^{n-k}(F_h^{(n)}(b))^{k-1}f_h^{(n)}(b)\left((k-1)v-kb+w_h\right)=0,$$

from which we can solve for the bid distribution $F_h^{(n)}(b)$. For instance, with three bidders (i.e., $n+1=3$), we have

$$F_h^{(2)}(b) = \frac{2(1-\rho)(b-w_l)}{\rho(v+w_h-2b)},$$

and setting $F_h(\bar{b}^{(2)})=1$ implies that the upper bound of the support, $\bar{b}^{(2)}$, is given by

$$\bar{b}^{(2)} = (1-\rho)w_l + \frac{\rho w_h}{2} + \frac{\rho v}{2}.$$

Since we must have $\bar{b} \le w_h$, therefore $F_h^{(2)}$ is an equilibrium only if

$$w_h \ge \frac{2(1-\rho)}{2-\rho}w_l + \frac{\rho}{2-\rho}v.$$

This corresponds to our analysis for the three-bidder case in Section 3. With four bidders (i.e., $n+1=4$), we have

$$F_h^{(3)}(b) = \frac{-3(1-\rho)(v+w_h-2b)+(1-\rho)\sqrt{9(v+w_h-2b)^2-12(w_l-b)(2v+w_h-3b)}}{2\rho(2v+w_h-3b)}.$$

Note that $F_h^{(3)}(w_l)=0$, and $F_h^{(3)}(\bar{b}^{(3)})=1$ implies that

$$\bar{b}^{(3)} = (1-\rho)^2 w_l + \rho(3-2\rho)\frac{w_h}{3} + \rho(3-\rho)\frac{v}{3}.$$

Then $\bar{b}^{(3)} \le w_h$ if and only if

$$w_h \ge \frac{3(1-\rho)^2}{3-3\rho+2\rho^2}w_l + \frac{\rho(3-\rho)}{3-3\rho+2\rho^2}v.$$

More generally, for any n, let $\bar{b}^{(n)}$ be such that $F_h^{(n)}(\bar{b}^{(n)}) = 1$. Then we have

$$\bar{b}^{(n)} = (1-\rho)^{n-1} w_l + \frac{1}{n} \sum_{k=2}^{n} \binom{n}{k} \rho^{k-1} (1-\rho)^{n-k} ((k-1)v + w_h),$$

and $F_h^{(n)}$ is an equilibrium only if

$$w_h \geq \frac{n(1-\rho)^{n-1} w_l + \sum_{k=2}^{n} \binom{n}{k} \rho^{k-1} (1-\rho)^{n-k} (k-1)v}{n - \sum_{k=2}^{n} \binom{n}{k} \rho^{k-1} (1-\rho)^{n-k}} \equiv \bar{w}_h^{(n)}.$$

That is, for each n we can identify a cutoff value $\bar{w}_h^{(n)}$ for the high budget level such that if the high budget bidders' budget is above this cutoff then they bid according to the atomless distribution $F_h^{(n)}$ in the first auction.

Next suppose that all other high budget bidders bid w_h with probability one. Consider a high budget bidder's incentive to bid w_h or less than w_h. The bidder's payoff from bidding w_h is:

$$\Pi_h(w_h) = (1-\rho)^n (v - w_l) + n\rho(1-\rho)^{n-1} \left(\frac{v - w_h}{2} + \frac{v - w_l}{2} \right) + \sum_{k=2}^{n} \binom{n}{k} \rho^k (1-\rho)^{n-k} \frac{2(v - w_h)}{k+1}.$$

On the other hand, the bidder's payoff from bidding $w_{h-} \in (w_l, w_h)$ is:

$$\Pi_h(w_{h-}) = (1-\rho)^n (v - w_l) + n\rho(1-\rho)^{n-1} (v - w_l) + \sum_{k=2}^{n} \binom{n}{k} \rho^k (1-\rho)^{n-k} \frac{(v - w_h)}{k}.$$

Then we have

$$\Pi_h(w_{h-}) - \Pi_h(w_h) = n\rho(1-\rho)^{n-1} \left(\frac{w_h - w_l}{2} \right) - \sum_{k=2}^{n} \binom{n}{k} \rho^k (1-\rho)^{n-k} \frac{(k-1)(v - w_h)}{k(k+1)} > 0$$

if and only if

$$w_h > \frac{\frac{n}{2}\rho(1-\rho)^{n-1}}{\frac{n}{2}\rho(1-\rho)^{n-1} + \sum_{k=2}^{n} \binom{n}{k} \rho^k (1-\rho)^{n-k} \frac{(k-1)}{k(k+1)}} w_l + \frac{\sum_{k=2}^{n} \binom{n}{k} \rho^k (1-\rho)^{n-k} \frac{(k-1)}{k(k+1)}}{\frac{n}{2}\rho(1-\rho)^{n-1} + \sum_{k=2}^{n} \binom{n}{k} \rho^k (1-\rho)^{n-k} \frac{(k-1)}{k(k+1)}} v \equiv \underline{w}_h^{(n)}.$$

Note that $\bar{w}_h^{(n)} > \underline{w}_h^{(n)}$ for each $n \geq 2$. This gives us another cutoff $\underline{w}_h^{(n)}$ of the high budget level such that if the high budget bidders' budget is below this cutoff then they bid their entire budget in the first auction.

For example, with three bidders (i.e., $n + 1 = 3$), we have

$$\underline{w}_h^{(2)} = \frac{6(1-\rho)}{6-5\rho} w_l + \frac{\rho}{6-5\rho} v;$$

with four bidders (i.e., $n + 1 = 4$), we have

$$\underline{w}_h^{(3)} = \frac{9(1-\rho)^2}{9(1-\rho)^2 + \rho(2-\rho)} w_l + \frac{\rho(2-\rho)}{9(1-\rho)^2 + \rho(2-\rho)} v.$$

Note that for any $\rho \in (0, 1)$,

$$\frac{\rho(2-\rho)}{9(1-\rho)^2 + \rho(2-\rho)} > \frac{\rho}{6-5\rho}.$$

Therefore, $\underline{w}_h^{(3)} > \underline{w}_h^{(2)}$. See Example 5.4 for detailed comparisons between the three and four bidders cases.

In fact, the next result (Proposition 5.2) shows that the sequences $(\underline{w}_h^{(n)})_n$ and $(\bar{w}_h^{(n)})_n$ are both increasing. This supports the intuitive idea that with more bidders and hence more competition, it is more likely for high budget bidders to bid their budget rather than to randomize over an interval of bids. An immediate corollary (Corollary 5.3) then is that prices will decline with enough competition.

Proposition 5.2. *The sequences of cutoffs, $(\underline{w}_h^{(n)})_n$ and $(\bar{w}_h^{(n)})_n$, are both increasing in n. Moreover, if $2w_l > v$, then we have $\lim_{n\to\infty} \underline{w}_h^{(n)} = \lim_{n\to\infty} \bar{w}_h^{(n)} = v$.*

Proof. We first show that the sequence $(\underline{w}_h^{(n)})_n$ is increasing in n. For any n, we know that when $w_h = \underline{w}^{(n)}$, a bidder with high budget is indifferent between bidding w_h or w_{h-}. Now suppose there are $n + 2$ bidders. By the definition of $\underline{w}_h^{(n)}$, the payoff difference for a high budget bidder between bidding $\underline{w}_h^{(n)}$ and $\underline{w}_{h-}^{(n)}$, when all other high budget bidders bid $\underline{w}_h^{(n)}$, can be written as

$$
\begin{aligned}
&\Pi_h^{(n+1)}(\underline{w}_h^{(n)}) - \Pi_h^{(n+1)}(\underline{w}_{h-}^{(n)}) = (1-\rho)\left[\Pi_h^{(n)}(\underline{w}_h^{(n)}) - \Pi_h^{(n)}(\underline{w}_{h-}^{(n)})\right] \\
&\quad + \rho\left[(1-\rho)^n\left(\frac{w_l - \underline{w}_h^{(n)}}{2}\right) + \sum_{k=1}^{n}\binom{n}{k}\rho^k(1-\rho)^{n-k}\frac{k}{(k+1)(k+2)}(v - \underline{w}_h^{(n)})\right] \\
&\quad = \rho\left[(1-\rho)^n\left(\frac{w_l - \underline{w}_h^{(n)}}{2}\right) + \sum_{k=1}^{n}\binom{n}{k}\rho^k(1-\rho)^{n-k}\frac{k}{(k+1)(k+2)}(v - \underline{w}_h^{(n)})\right] \\
&\quad = \rho\left[(1-\rho)^n\left(\frac{w_l - \underline{w}_h^{(n)}}{2}\right) + \sum_{k=1}^{n}\binom{n}{k}\rho^k(1-\rho)^{n-k}\frac{k}{(k+1)(k+2)}(v - \underline{w}_h^{(n)})\right. \\
&\qquad\qquad + (1-\rho)^n\left(\frac{\underline{w}_h^{(n)} - w_l}{2}\right) \\
&\qquad\qquad \left. - \sum_{k=1}^{n-1}\frac{(n-1)!}{(n-1-k)!\,(k+1)!}\rho^k(1-\rho)^{n-k}\frac{k}{(k+1)(k+2)}(v - \underline{w}_h^{(n)})\right] \\
&\quad = \rho\left[\sum_{k=1}^{n}\binom{n}{k}\rho^k(1-\rho)^{n-k}\frac{k}{(k+1)(k+2)}\right. \\
&\qquad\qquad \left. - \sum_{k=1}^{n-1}\frac{(n-1)!}{(n-1-k)!\,(k+1)!}\rho^k(1-\rho)^{n-k}\frac{k}{(k+1)(k+2)}\right](v - \underline{w}_h^{(n)}) > 0,
\end{aligned} \tag{6}
$$

where the third equality uses the fact that

$$
\begin{aligned}
0 &= n\rho(1-\rho)^{n-1}\left(\frac{\underline{w}_h^{(n)} - w_l}{2}\right) - \sum_{k=2}^{n}\binom{n}{k}\rho^k(1-\rho)^{n-k}\frac{(k-1)(v - \underline{w}_h^{(n)})}{k(k+1)} \\
&= (1-\rho)^n\left(\frac{\underline{w}_h^{(n)} - w_l}{2}\right) - \sum_{k=2}^{n}\frac{(n-1)!}{(n-k)!\,k!}\rho^{k-1}(1-\rho)^{n-k+1}\frac{k-1}{k(k+1)}(v - \underline{w}_h^{(n)}) \\
&= (1-\rho)^n\left(\frac{\underline{w}_h^{(n)} - w_l}{2}\right) - \sum_{k=1}^{n-1}\frac{(n-1)!}{(n-1-k)!\,(k+1)!}\rho^k(1-\rho)^{n-k}\frac{k}{(k+1)(k+2)}(v - \underline{w}_h^{(n)}),
\end{aligned}
$$

and the inequality follows from

$$
\binom{n}{k} = \frac{n!}{(n-k)!\,k!} > \frac{(n-1)!}{(n-1-k)!\,(k+1)!}
$$

Since for each n we have $\Pi_h^{(n)}(w_{h-}) > \Pi_h^{(n)}(w_h)$ if and only if $w_h \geq \underline{w}_h^{(n)}$, it follows that $\underline{w}_h^{(n+1)} > \underline{w}_h^{(n)}$. By a similar argument, the sequence $(\underline{w}_h^{(n)})_n$ is also increasing in n.

Finally, we show that $\lim_{n\to\infty} \underline{w}_h^{(n)} \equiv \underline{w}_h^{\infty} = v$. Recall that

$$\underline{w}_h^{(n)} = \frac{\frac{n}{2}\rho(1-\rho)^{n-1}}{\frac{n}{2}\rho(1-\rho)^{n-1} + \sum_{k=2}^{n}\binom{n}{k}\rho^k(1-\rho)^{n-k}\frac{(k-1)}{k(k+1)}} w_l + \frac{\sum_{k=2}^{n}\binom{n}{k}\rho^k(1-\rho)^{n-k}\frac{(k-1)}{k(k+1)}}{\frac{n}{2}\rho(1-\rho)^{n-1} + \sum_{k=2}^{n}\binom{n}{k}\rho^k(1-\rho)^{n-k}\frac{(k-1)}{k(k+1)}} v.$$

It suffices to show that

$$\lim_{n\to\infty} \frac{\frac{n}{2}\rho(1-\rho)^{n-1}}{\sum_{k=2}^{n}\binom{n}{k}\rho^k(1-\rho)^{n-k}\frac{(k-1)}{k(k+1)}} = 0.$$

Since $\frac{(k-1)}{k(k+1)}$ is decreasing in k, we have

$$\frac{\frac{n}{2}\rho(1-\rho)^{n-1}}{\sum_{k=2}^{n}\binom{n}{k}\rho^k(1-\rho)^{n-k}\frac{(k-1)}{k(k+1)}} \leq \frac{\frac{n}{2}\rho(1-\rho)^{n-1}}{\frac{(n-1)}{n(n+1)}\sum_{k=2}^{n}\binom{n}{k}\rho^k(1-\rho)^{n-k}}.$$

Since $\sum_{k=2}^{n}\binom{n}{k}\rho^k(1-\rho)^{n-k}$ converges to one as n goes to infinity, the result follows from the fact that

$$\lim_{n\to\infty} \frac{\frac{n}{2}\rho(1-\rho)^{n-1}}{\frac{(n-1)}{n(n+1)}} = 0.$$

□

The intuition of the above result is as follows. Consider the effect of adding one more bidder holding other bidders types fixed, so that there are one more opponent from an existing bidder's viewpoint. There are two cases to consider for this additional bidder: first, if he has low budget, then an existing bidder with high budget should still be indifferent between w_h and w_{h-}; second, if the additional bidder has high budget, then an existing bidder with high budget would no longer be indifferent between w_h and w_{h-}. This is because tie-breaking the first auction can be treated as a sequential procedure, that is, when all high budget bidders bid w_h, we can first consider the case where an existing bidder i is tied with the additional bidder who has high budget, if bidder i wins this tie, then bidder i's payoff is determined by tie-breaking with all the remaining bidders. Since bidding w_{h-} means that the bidder will lose the tie against the additional bidder who bids w_h, it follows that when $w_h = \underline{w}^{(n)}$, bidding w_{h-} is worse than w_h when the additional bidder bids w_h.

Corollary 5.3. *For any given w_l, w_h, ρ satisfying $w_l < w_h < 2w_l$ and $w_h < v$, there exists $\bar{N} \in \mathbb{N}$ such that if the number of bidders is larger than $\bar{N}$, then the prices decline in the sequential auctions.*

Finally, for intermediate budget levels, i.e., $w_h \in (\underline{w}_h^{(n)}, \bar{w}_h^{(n)})$, following the similar argument as in Section 3, we can show that the high budget bidders' equilibrium bid distribution in the first auction has support $(w_l, \hat{b}^{(n)}] \cup \{w_h\}$ and there is an atom at w_h with probability mass $\Delta_h^{(n)} \in (0,1)$, where $\hat{b}^{(n)}$ is given by

$$F_h^{(n)}(\hat{b}^{(n)}) = 1 - \Delta_h^{(n)},$$

with $\Delta_h^{(n)} \in (0,1)$ solving the equation

$$n\rho(1-\rho)^{n-1}\Delta_h \left(\frac{v-w_l}{2}\right) = \sum_{m=2}^{n} \binom{n}{m} \rho^m (1-\rho)^{n-m} \left[\sum_{k=1}^{m} \binom{m}{k} \Delta_h^k (1-\Delta_h)^{m-k} \frac{k-1}{k(k+1)}\right] (v-w_h).$$

Example 5.4. Here we compare the high budget bidders' equilibrium bid distributions in the first period for cases with three or four bidders. To fix ideas, we focus on parameters under which the equilibria have atomless distributions in both cases. Let $v = 12$, $v_l = 4$, $v_h = 7$ and $\rho = 0.25$. Figure 11 illustrates the bid distributions calculated above with three or four bidders. In particular, the high budget bidders bid more aggressively in the case with four bidders than in the case with three bidders.

5.3. **The Case where $\mathbf{w_h > 2w_l}$.** When the high budget level is more than two times the low budget level a new strategic consideration arises. Now, if a high budget bidder competes against two low budget budgets, she is sure to win both auctions. This in tern changes the functional forms of the payoff, and as such has to be treated separately from the earlier

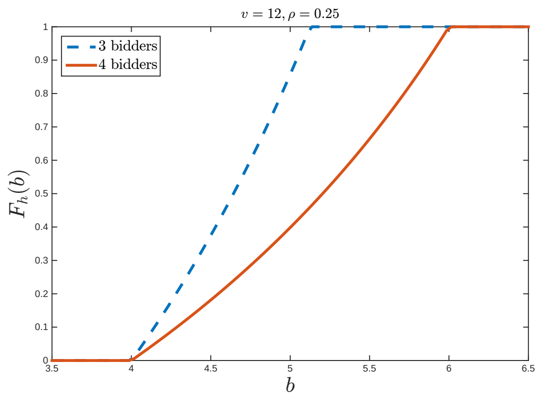

FIGURE 11. Bid distributions with three or four bidders

case.[10] In this section we outline the qualitative features of the equilibrium and provides sketches of the proofs of existence and uniqueness.

With a few modifications, Lemmas 3.2 to 3.8 apply in this case as well. In the appendix we sketch proofs for each of these. Lemma 3.8 implies that if $w_h > \underline{w}$, then in equilibrium the high budget bidder will play a mixed strategy in the first auction. The mixing will be continuous, as implied by Lemma 3.2. With this in mind, we can write the payoff function for a high budget bidder who bids $b < w_h$ as follows.

$$\begin{aligned}\Pi_h(b) &= (1-\rho)^2(v - w_l + v - w_l)\\ &\quad + 2\rho(1-\rho)\left[\int_{\underline{b}}^{b}(v-x)dF_h(x) + (1-F_h(b))(v - \max\{w_l, w_h - b\})\right]\\ &\quad + \rho^2\left[\int_{\underline{b}}^{b}(v-x)dF_h(x)^2 + (1-F_h(b)^2)\frac{v-w_h}{2}\right]\end{aligned}$$

Note that since $w_h > 2w_l$, the high budget bidder will win both auctions against low budget bidders. When competing against one high and one low budget bidder, her payoff in the second auction, in the event of losing the first, will depend on the bid she places in the first auction, as her bid in the first auction determines the price, and hence the remaining budget of the other high budget bidder. If $b < w_h - w_l$, and she loses the first auction, the second

[10]The case of $w_h = 2w_l$ is similar to the case above but leads to another complication of the possibilities of ties between high budget and low budget bidders in the second auction. We can solve this case as well, but abstract from it to focus on the more generic case of $w_h > 2w_l$.

highest bid, and hence the price, in second auction is $w_h - b$.[11] If $b \geq w_h - w_l$ and the bidder loses the first auction, then the second highest bid the second auction will be w_l. Given the threshold of $w_h - w_l$, we can derive the slope of the payoff function in two regions as

$$\frac{d\Pi_h(b)}{db} = \begin{cases} \begin{aligned} &2\rho(1-\rho)\left[(v-b)f_h(b) - f_h(b)(v-w_h+b) + (1-F_h(b))\right] \\ &\qquad + \rho^2\left[2(v-b)f_h(b)F_h(b) - f_h(b)F_h(b)(v-w_h)\right] \end{aligned}; & \text{if } b < w_h - w_l \\ \begin{aligned} &2\rho(1-\rho)\left[(v-b)f_h(b) - f_h(b)(v-w_l)\right] \\ &\qquad + \rho^2\left[2(v-b)f_h(b)F_h(b) - f_h(b)F_h(b)(v-w_h)\right] \end{aligned}; & \text{if } w_h > b \geq w_h - w_l \end{cases}$$

Equating to zero, we get the functional forms for the distribution function over two regions.

$$F_h(b) = \begin{cases} \dfrac{2(1-\rho)\left((2b-w_h)f_h(b) - 1\right)}{\rho(v+w_h-2b)f_h(b) - 2(1-\rho)} \equiv \hat{F}_h(b); & \text{if } b < w_h - w_l \\ \dfrac{2(1-\rho)(b-w_l)}{\rho(v+w_h-2b)} \equiv \bar{F}_h(b); & \text{if } w_h > b \geq w_h - w_l \end{cases} \tag{7}$$

If the bidder is mixing over a set of bids, then she must be following the above distribution function. Note that the $\bar{F}_h$ is exactly the same distribution function as in the case $w_h \leq 2w_l$. Let $\hat{f}_h = \hat{F}_h'$ and $\bar{f}_h = \bar{F}_h'$. Let $\underline{b} = \inf \operatorname{supp} F_h$ and $\bar{b} = \sup \operatorname{supp} F_h$. The following lemma makes some observations about the left end of the support.

Lemma 5.5. *It must be the case that $\underline{b} > \frac{w_h}{2}$ and $\underline{b} < w_h - w_l$.*

Proof. Suppose $\underline{b} \leq \frac{w_h}{2}$. Since $w_h > 2w_l$, this implies $\underline{b} < w_h - w_l$. Therefore for $b \in (\underline{b}, \underline{b} + \varepsilon)$ for $\varepsilon > 0$ and small, the probability density at b is given by $\hat{f}_h(b)$. Note that $\hat{f}_h(b)$ is negative for $b \leq \frac{w_h}{2}$, a contradiction.

Now suppose $\underline{b} \geq w_h - w_l$. Then the bid distribution must be given by $\bar{F}_h$. However, in this case $\bar{F}_h(\underline{b}) = 0$ implies $\underline{b} = w_l$, a contradiction to $\underline{b} \geq w_h - w_l$, since $w_h > 2w_l$. □

Lemma 5.5 shows that if $w_h > \underline{w}$ and an equilibrium exists, then the equilibrium bid distribution must include the function $\hat{F}_h$. We need to make sure that $\hat{F}_h$ is indeed a distribution function. The next lemma establishes this.

[11] Recall $b > w_l$, due to Lemma 3.7.

Lemma 5.6. *Suppose that* $w_h \geq 2w_l$. *If* $\hat{F}(\bar{b}) = 1$, *then* $\bar{b} = \frac{1}{2}w_h + \frac{\rho}{4-2\rho}v$ *and* $\hat{F}_h(\cdot)$ *is increasing for* $b \leq \bar{b}$.

Proof. Consider the differential equation

$$\hat{F}_h(b) = \frac{2(1-\rho)((2b-w_h)\hat{f}_h(b)-1)}{\rho(v+w_h-2b)\hat{f}_h(b)-2(1-\rho)} \tag{8}$$

Suppose there exists a $\bar{b} < w_h$ such that $\hat{F}_h(\bar{b}) = 1$. Rewrite the above equation as

$$\hat{f}_h(b) = \frac{2(1-\rho)(1-\hat{F}_h(b))}{2(1-\rho)(2b-w_h)-\rho(v+w_h-2b)\hat{F}_h(b)}. \tag{9}$$

Since $\hat{f}_h(b)$ can take value 0 only at $\bar{b}$ with $1-\hat{F}_h(\bar{b}) = 0$ and at $\bar{b}$ the denominator of $\hat{f}_h(b)$ is also zero, by L'Hôpital's rule we have

$$\hat{f}_h(\bar{b}) = \frac{-2(1-\rho)\hat{f}_h(\bar{b})}{4(1-\rho)-\rho(v+w_h-2\bar{b})\hat{f}_h(\bar{b})+2\rho},$$

which implies that either $\hat{f}_h(\bar{b}) = 0$ or $\hat{f}_h(\bar{b}) = \frac{6-4\rho}{\rho(v+w_h-2\bar{b})} > 0$. If $\hat{f}_h(\bar{b}) = 0$, since the slope of the function

$$g(b) = \frac{2(1-\rho)(2b-w_h)}{\rho(v+w_h-2b)}$$

at $\bar{b}$ is positive, it follows that the denominator of $\hat{f}_h(b)$, $2(1-\rho)(2b-w_h)-\rho(v+w_h-2b)\hat{F}_h(b)$, is less than zero at $\bar{b}-\varepsilon$ for sufficiently small $\varepsilon > 0$. Thus, if $\hat{F}_h(\bar{b}-\varepsilon) < 1$, then $\hat{f}_h(\bar{b}-\varepsilon) < 0$, which implies that $\hat{F}_h(b) > 1$ for $b < \bar{b}$, a contradiction; if $\hat{F}_h(\bar{b}-\varepsilon) > 1$, then $\hat{f}_h(\bar{b}-\varepsilon) > 0$, which which implies that $\hat{F}_h(b) < 1$ for $b < \bar{b}$, again a contradiction. Hence, we have

$$\hat{f}_h(\bar{b}) = \frac{6-4\rho}{\rho(v+w_h-2\bar{b})} > 0,$$

if $\hat{F}_h(\bar{b}) = 1$. Therefore, from (8), $\hat{F}_h(\bar{b}) = 1$ implies that

$$\bar{b} = \frac{1}{2}w_h + \frac{\rho}{4-2\rho}v \tag{10}$$

Moreover, since $f_h(\bar{b}) > 0$, we have $f_h(b) \neq 0$ for any b, it follows that $\hat{F}_h'(b) = \hat{f}_h(b) > 0$ for any $b \leq \bar{b}$. □

Using Lemma 5.6, we can say that $\hat{F}_h$ is indeed a distribution function. If the equilibrium is to be completely characterized by $\hat{F}_h$ then it must be the case that $\bar{b} \le w_h - w_l$. Using this restriction, the next proposition states that for some parameters the equilibrium can be completely characterized by $\hat{F}_h$.

Proposition 5.7. *If $w_h \ge 2w_l + \frac{\rho}{2-\rho}v$, then there exists a unique symmetric equilibrium where the low budget bidders always bid their budgets and the high budget bidders mix continuously in the first auction using $\hat{F}_h$.*

Proof. Suppose $w_h \ge \hat{w} \equiv 2w_l + \frac{\rho}{2-\rho}v$. Then $\bar{b} \le w_h - w_l$. Hence if the bidder is mixing in the first auction she must be using the bid distribution $\hat{F}_h$. By construction $\Pi_h(b)$ is the same for all $b \in [\underline{b}, \bar{b}]$. If a bidder bids strictly higher than $\bar{b}$ her payoff does not change due to the nature of the second price auction. If she bids strictly below $\underline{b}$, then she is lowering the price in first auction in the case there is one low budget and one high budget bidder and the other high budget bidder wins, which will surely happen since she is bidding below $\underline{b}$. This action leaves a higher budget for the second auction for the other high budget bidder, which in tern increases the price she will pay in the second auction, thereby reducing her payoff. Hence $\hat{F}_h$ is the unique equilibrium strategy. □

Since we do not have a closed form for the distribution function, we take aid of a numerical exercise to illustrate the equilibrium in the next example.

Example 5.8. Consider $v = 10, w_l = 2$ and $\rho = 0.5$. In this case $\hat{w} = 7.33$. Therefore for any $w_h \in (\hat{w}, v)$ the equilibrium bid distribution is given by $\hat{F}_h(\cdot)$. Note that $\bar{b}$ can be thought of as a function of w_h. Therefore, we can calculate the value of $\bar{b}(w_h)$ from equation (10), where it is a function of the high budget level. Also, $\hat{F}_h(\bar{b}(w_h)) = 1$. Therefore, given a high budget level, w_h, and the boundary condition $\bar{b}(w_h), \hat{F}_h(\bar{b}(w_h))$, we can numerically solve for the unique bid distribution, using the ordinary differential equation (8). In figure 12, we show two such bid distributions, with high budget levels $w_h^1 = 9 > 8 = w_h^2$. Note that $w_h^1, w_h^2 \in (\hat{w}, v)$.

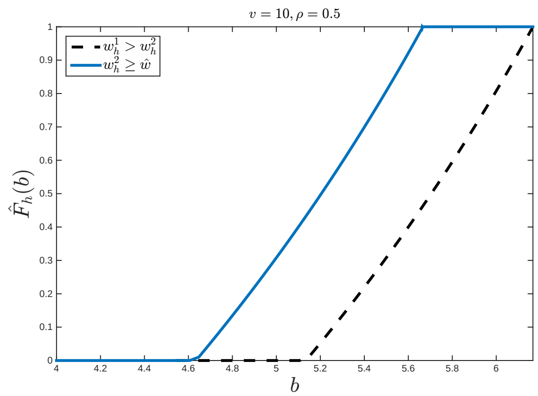

FIGURE 12. $\hat{F}_h$ for budget level w_h^1, w_h^2

From the figure it is clear that the equilibrium bid distribution of an auction where the high budget level is w_h^1, stochastically dominates the bid distribution with a high budget level $w_h^2 < w_h^2$. This immediately implies that the expected bid placed by a high budget level is also rising in the budget levels.

Using numerical integration we calculate the expected bid submitted by high budget bidders in the first auction and present the results in table 2. In the table we also calculate relative aggression, $\alpha(\cdot)$, as previously defined. Even in the case $w_h > 2w_l$ (and $w_h > \hat{w}$), we can see that relative aggression declines as the budget level increases.

w_h	$\mathbb{E}(b \mid w_h)$	$\alpha(w_h)$
7.5	4.95	0.661
8	5.20	0.650
8.5	5.45	0.641
9	5.70	0.634
9.5	5.95	0.627
10	6.20	0.620

TABLE 2. Expected bids and relative aggression

Till now the qualitative features of the equilibrium for the case $w_h > 2w_l$ have been identical to the case $w_h \leq 2w_l$. For w_h below a threshold ($\underline{w}$) there exists a unique equilibrium in pure strategies. For w_h above some thresholds ($\bar{w}$ in the case $w_h \leq 2w_l$ and $\hat{w}$ in the case $w_h > 2w_l$), the unique symmetric equilibrium involves continuous mixing for the high budget level. Next, we consider the remaining case $w_h \in (\underline{w}, \hat{w})$. In this case, (unlike $w_h \leq 2w_l$ and $w_h \in (\underline{w}, \bar{w})$), equilibrium may fail to exist. If it exists it takes a similar qualitative form to the earlier case. We begin by establishing that an equilibrium where the bid distribution includes both $\bar{F}_h$ and $\hat{F}_h$ distribution functions cannot exist.

Lemma 5.9. *Suppose that $w_h \geq 2w_l$ and $w_h \in (\underline{w}, \hat{w})$. Then $\hat{F}_h(w_h - w_l) < \bar{F}_h(w_h - w_l)$.*

Proof. Since $w_h < \hat{w} = 2w_l + \frac{\rho}{2-\rho}v$, we have

$$\bar{F}_h(w_h - w_l) = \frac{2(1-\rho)(w_h - 2w_l)}{\rho(v + 2w_l - w_h)} < 1.$$

Suppose $\hat{f}_h(w_h - w_l) < \infty$. Since

$$\hat{F}_h(w_h - w_l) = \frac{2(1-\rho)((w_h - 2w_l)\hat{f}_h(w_h - w_l) - 1])}{\rho(v + 2w_l - w_h)\hat{f}_h(w_h - w_l) - 2(1-\rho)} \leq 1,$$

if $\hat{F}_h(w_h - w_l) > \bar{F}_h(w_h - w_l)$ and $\hat{F}_h(w_h - w_l) \neq 1$, then $\hat{f}_h(w_h - w_l) < 0$, which is impossible; if $\hat{F}_h(w_h - w_l) = 1$, then $\hat{f}_h(w_h - w_l) = 0$, which contradicts Lemma 5.6. Therefore, we have $\hat{F}_h(w_h - w_l) \leq \bar{F}_h(w_h - w_l)$. Suppose $\hat{F}_h(w_h - w_l) = F_h(w_h - w_l)$. Then $\lim_{b \to w_h - w_l} \hat{f}_h(b) = \infty$, which implies that $2(1-\rho)(2b - w_h) - \rho(v + w_h - 2b)F_h(b) < 0$ for $b = w_h - w_l + \varepsilon$ and $\varepsilon > 0$ is sufficiently small. Since $\hat{F}_h(w_h - w_l) < 1$, it follows that $\lim_{b \downarrow (w_h - w_l)} \hat{f}_h(b) < 0$, which contradicts the fact that $\hat{F}_h(\cdot)$ is differentiable at $w_h - w_l$. Hence, we must have $\hat{F}_h(w_h - w_l) < \bar{F}_h(w_h - w_l)$. □

Since there can be no atom in the bid distribution, an equilibrium that contains both $\bar{F}_h$ and $\hat{F}_h$ cannot exist. We are now in a position to state our final result that characterizes the equilibrium, whenever it exists, in the case $w_h \in (\underline{w}, \hat{w})$ and $w_h > 2w_l$.

Proposition 5.10. *Suppose that $w_h > 2w_l$ and $w_h \in (\underline{w}, \hat{w})$. If there exists a symmetric equilibrium, then the high budget bidder's bid distribution F_h takes the following form: there* *exists* $\underline{b}, \hat{b}$ *with* $\underline{b} < \hat{b}$, $\underline{b} \geq \frac{w_h}{2}$ *and* $\hat{b} \leq w_h - w_l$ *such that* $F_h(b) = \hat{F}_h(b)$ *for* $b \in [\underline{b}, \hat{b}]$*;* $F_h(b) = \hat{F}_h(\hat{b})$ *for* $b \in [\hat{b}, w_h)$*; and* $F_h(b) = 1$ *for* $b \in [w_h, \infty)$*. Furthermore, there exists a symmetric equilibrium if and only if the equation*

$$3\rho(v - w_h) - 6(1-\rho)\left(2\hat{b} - w_h + w_l\right)2\rho(v - w_h) = \frac{\left[\rho v - (2-\rho)(2\hat{b} - w_h)\right]\hat{f}_h(\hat{b})}{\rho(v + w_h - 2\hat{b})\hat{f}_h(\hat{b}) - 2(1-\rho)} \tag{11}$$

has a solution $\hat{b} \in [\underline{b}, w_h - w_l]$*.*

Proof. By Lemmas 5.6 and 5.9, and the fact that there cannot be an atom at a level less than w_h in the high budget bidder's bid distribution, if a symmetric equilibrium exists, then the support of the high budget bidder's bid distribution $\hat{F}_h$ is $[\underline{b}, \hat{b}] \cup \{w_h\}$, where $\underline{b} \geq w_h/2$ and $\hat{b} \leq w_h - w_l$, and the only atom is w_h with probability $\Delta \in (0,1)$. For the bidder to be indifferent among all bids in $[\underline{b}, \hat{b}]$, it must be that $F_h(b) = \hat{F}_h(b)$, for any $b \in [\underline{b}, \hat{b}]$. Then $\Delta = 1 - \hat{F}_h(\hat{b})$.

The payoff from bidding $\hat{b}$ is

$$\Pi_h(\hat{b}) = 2(1-\rho)^2(v - w_l) + 2\rho(1-\rho)\left[\int_{\underline{b}}^{\hat{b}}(v-x)d\hat{F}_h(x) + (1 - \hat{F}_h(\hat{b}))(v - (w_h - \hat{b}))\right]$$
$$+ \rho^2\left[\int_{\underline{b}}^{\hat{b}}(v-x)d\hat{F}_h^2(x) + (1 - \hat{F}_h^2(\hat{b}))\frac{v - w_h}{2}\right].$$

The payoff from bidding w_h is

$$\Pi_h(w_h) = 2(1-\rho)^2(v - w_l) + 2\rho(1-\rho)\left[\int_{\underline{b}}^{\hat{b}}(v-x)d\hat{F}_h(x) + (1 - \hat{F}_h(\hat{b}))\left(\frac{v - w_h}{2} + \frac{v - w_l}{2}\right)\right]$$
$$+ \rho^2\left[\int_{\underline{b}}^{\hat{b}}(v-x)d\hat{F}_h^2(x) + 2\Delta(1-\Delta)\left(\frac{v - w_h}{2} + \frac{v - w_h}{4}\right)\right.$$
$$\left. + \Delta^2\left(\frac{v - w_h}{3} + \frac{v - w_h}{3}\right)\right].$$

If $\hat{F}_h$ is an equilibrium, then it must be that $\Pi_h(\hat{b}) = \Pi_h(w_h)$, which implies

$$(1-\rho)\left(2\hat{b} - w_h + w_l\right) = \rho\left(2(1-\Delta)\left(\frac{v - w_h}{4}\right) + \Delta\left(\frac{v - w_h}{6}\right)\right).$$

Solving this equation for Δ, we have

$$\Delta = \frac{3\rho(v - w_h) - 6(1-\rho)\left(2\bar{b} - w_h + w_l\right)}{2\rho(v - w_h)} \equiv \xi(\bar{b}).$$

Since $\Delta = 1 - \hat{F}_h(\hat{b})$, we also have

$$\Delta = 1 - \hat{F}_h(\hat{b}) = \frac{\left[\rho v - (2-\rho)(2\hat{b} - w_h)\right] \hat{f}_h(\hat{b})}{\rho(v + w_h - 2\hat{b}) f_h(\hat{b}) - 2(1-\rho)}$$

Note that the function $\hat{F}_h$ is unique given the boundary condition

$$\hat{F}_h(\bar{b}) = 1,$$

where $\bar{b} = \frac{1}{2}w_h + \frac{\rho}{4-2\rho}v$. And we have $\hat{F}_h(\underline{b}) = 0$ for some $\underline{b} \in \left(\frac{w_h}{2}, w_h - w_l\right)$. Therefore, if equation (11) has a solution $\hat{b} \in [\underline{b}, w_h - w_l]$, then there exists a symmetric equilibrium and the equilibrium bid distribution F_h is: $F_h(b) = \hat{F}_h(b)$ for $b \in [\underline{b}, \hat{b}]$, $F_h(b) = \hat{F}_h(\hat{b})$ for $b \in [\hat{b}, w_h)$, and $F_h(b) = 1$ for $b \in [w_h, \infty)$; otherwise, there is no symmetric equilibrium.

□

6. Conclusion

In this paper we found the unique symmetric equilibrium of an auction game where two units are sold in second price auctions sequentially to bidders who are budget constrained and are uncertain of each other's budgets. We were able to unearth strategic considerations budgets bring into a sequential auction setting. These strategic considerations led to many unique features of the equilibrium. Chief among these features was that prices may decline in expectation, an empirical phenomenon that has been observed in sequential auctions of identical items.

Appendix A

In this appendix we collect the proofs of lemmas 3.2-3.11.

Proof of Lemma 3.2: We prove this lemma in three steps. First, we will show that $F_l(\cdot)$ is atom-less. Next we show that $F_h(b)$ is continuous at $b < w_l$. Finally, we show that $F_h(b)$ is continuous at $w_h > b > w_l$.

Step 1: Suppose there exists $b < w_l$ such that $\lim_{\varepsilon \to 0} F_l(b) - F_l(b - \varepsilon) = \Delta_l > 0$ and $\lim_{\varepsilon \to 0} F_h(b) - F_h(b - \varepsilon) = \Delta_h \geq 0$. In the first part of the proof we will show that $\Delta_l = 0$.

We have to further divide this step into two cases, depending on the value of b relative to $w_h - w_l$ as the payoff function of a bidder changes around this value. To see this, suppose a low budget bids $b < w_h - w_l$ in the first auction. Consider the case where this bidder is competing with one high and one low budget bidder. If b is the second highest bid, then the bidder can not win the second auction, as the high budget bidder will have more budget than this bidder in the second auction for sure. Now suppose, $b > w_h - w_l$. Again, consider the case where this bidder's competitors are l and h. If b is the second highest bid, then the bidder may win the second auction if the highest bid in the first auction is placed by the high budget bidder. The above reasoning implies that depending on the value of b, $\Pi_l^{lh}(b)$ can take different forms. This reasoning also applies to other parts of the payoff function.

Case (i): $b \geq w_h - w_l$

Any bidder's expected payoff from bidding b in the first auction can be split up into three parts, based on the bidder's competitors as stated in (1). The payoff to a low budget bidder from bidding b are stated below.

$$\Pi_l^{ll}(b) = (1-\rho)^2 \Bigg[(F_l(b) - \Delta_l)^2 \mathbb{E}[v - \text{second highest bid}] + 2(F_l(b) - \Delta_l)\Delta_l \left(\frac{v-b}{2} + \frac{v-w_l}{4} \right)$$

$$+ \Delta_l^2 \left(\frac{v-b}{3} + \frac{2}{3}\left(\frac{v-w_l}{2} \right) \right) + (1 - F_l(b)^2) \left(\frac{v-w_l}{2} \right) \Bigg]$$

$$\Pi_l^{lh}(b) = 2\rho(1-\rho)\Bigg[(F_l(b)-\Delta_l)(F_h(b)-\Delta_h)\mathbb{E}[v - \text{second highest bid}] + (F_l(b)-\Delta_l)\Delta_h\left(\frac{v-b}{2} - \frac{v-w_l}{2}\right) + (F_l(b)-\Delta_l)(1-F_h(b))\left(\frac{v-w_l}{2}\right) + \Delta_l(F_h(b)-\Delta_h)\left(\frac{v-b}{2}\right) + \Delta_l\Delta_h\left(\frac{v-b}{3}+\frac{1}{3}\left(\frac{v-w_l}{2}\right)\right) + \Delta_l(1-F_h(b))\left(\frac{v-w_l}{2}\right) + (1-F_l(b))(F_h(b)-\Delta_h)(0) + (1-F_(b))\Delta_h(0) + (1-F_l(b))(1-F_h(b))(\text{Pr h wins first auction})\left(\frac{v-w_l}{2}\right)\Bigg]$$

$$\Pi_h^{hh}(b) = \rho^2\Bigg[(F_h(b)-\Delta_h)^2\mathbb{E}[v - \text{second highest bid}] + 2(F_h(b)-\Delta_h)\Delta_h\left(\frac{v-b}{2}\right) + \Delta_h^2\left(\frac{v-b}{3}\right) + \left(1-F_h(b)^2\right)(0)\Bigg]$$

Now, suppose the bidder decides to bid $b_+ = b+\varepsilon$, where $\varepsilon > 0$, but very small. Then the payoff from bidding b_+ is approximately equal to the sum of the following three functions.

$$\Pi_l^{ll}(b_+) \approx (1-\rho)^2\Bigg[(F_l(b)-\Delta_l)^2\mathbb{E}[v - \text{second highest bid}] + 2(F_l(b)-\Delta_l)\Delta_l(v-b) + \Delta_l^2(v-b) + (1-F_l(b)^2)\left(\frac{v-w_l}{2}\right)\Bigg]$$

$$\Pi_l^{lh}(b_+) \approx 2\rho(1-\rho)\Bigg[(F_l(b)-\Delta_l)(F_h(b)-\Delta_h)\mathbb{E}[v - \text{second highest bid}] + (F_l(b)-\Delta_l)\Delta_h(v-b) + (F_l(b)-\Delta_l)(1-F_h(b))\left(\frac{v-w_l}{2}\right) + \Delta_l(F_h(b)-\Delta_h)(v-b) + \Delta_l\Delta_h(v-b) + \Delta_l(1-F_h(b))\left(\frac{v-w_l}{2}\right) + (1-F_l(b))(F_h(b)-\Delta_h)(0) + (1-F_(b))\Delta_h(0) + (1-F_l(b))(1-F_h(b))(\text{Pr h wins first auction})\left(\frac{v-w_l}{2}\right)\Bigg]$$

$$\Pi_h^{hh}(b_+) \approx \rho^2\left[(F_h(b)-\Delta_h)^2\mathbb{E}[v - \text{second highest bid}] + 2(F_h(b)-\Delta_h)\Delta_h(v-b) + \Delta_h^2(v-b) + (1-F_h(b)^2)(0)\right]$$

Clearly, $\Pi_l^{ll}(b) < \Pi_l^{ll}(b_+), \Pi_l^{lh}(b) < \Pi_l^{lh}(b_+)$ and $\Pi_l^{hh}(b) < \Pi_l^{hh}(b_+)$, since $b < w_l$.[12] Therefore $\Pi_l(b) < \Pi_l(b_+)$. Hence $\Delta_l = 0$ as bids slightly above b give a better payoff.

[12] A low budget bidder cannot bid more than her budget.

Case (ii): $b < w_h - w_l$

The proof for this uses an identical technique as above with slight modifications to the payoff functions. For parsimony we write the complete payoff function below.

$$\begin{aligned}\Pi_l(b) = (1-\rho)^2 \Bigg[&(F_l(b)-\Delta_l)^2\mathbb{E}[v - \text{second highest bid}] + 2(F_l(b)-\Delta_l)\Delta_l\left(\frac{v-b}{2}+\frac{v-w_l}{4}\right)\\ &+ \Delta_l^2\left(\frac{v-b}{3}+\frac{2}{3}\left(\frac{v-w_l}{2}\right)\right) + (1-F_l(b)^2)\left(\frac{v-w_l}{2}\right)\Bigg]\\ + 2\rho(1-\rho)\Bigg[&(F_l(b)-\Delta_l)(F_h(b)-\Delta_h)\mathbb{E}[v - \text{second highest bid}]\\ &+ (F_l(b)-\Delta_l)\Delta_h\left(\frac{v-b}{2}\right) + \Delta_l(F_h(b)-\Delta_h)\left(\frac{v-b}{2}\right) + \Delta_l\Delta_h\left(\frac{v-b}{3}\right)\Bigg]\\ + \rho^2\Bigg[&(F_h(b)-\Delta_h)^2\mathbb{E}[v - \text{second highest bid}] + 2(F_h(b)-\Delta_h)\Delta_h\left(\frac{v-b}{2}\right)\\ &+ \Delta_h^2\left(\frac{v-b}{3}\right)\Bigg]\end{aligned}$$

Now suppose the low budget bidder were to bid slightly more. Her payoff from such a bid will be

$$\begin{aligned}\Pi_l(b_+) \approx (1-\rho)^2 \Bigg[&(F_l(b)-\Delta_l)^2\mathbb{E}[v - \text{second highest bid}] + 2(F_l(b)-\Delta_l)\Delta_l(v-b)\\ &+ \Delta_l^2(v-b) + (1-F_l(b)^2)\left(\frac{v-w_l}{2}\right)\Bigg]\\ &+ 2\rho(1-\rho)\Big[(F_l(b)-\Delta_l)(F_h(b)-\Delta_h)\mathbb{E}[v - \text{second highest bid}]\\ &+ (F_l(b)-\Delta_l)\Delta_h\,(v-b) + \Delta_l(F_h(b)-\Delta_h)\,(v-b) + \Delta_l\Delta_h\,(v-b)\Big]\\ &+ \rho^2\Big[(F_h(b)-\Delta_h)^2\mathbb{E}[v - \text{second highest bid}] + 2(F_h(b)-\Delta_h)\Delta_h\,(v-b) + \Delta_h^2\,(v-b)\Big]\end{aligned}$$

Clearly, $\Pi_l(b_+) > \Pi_l(b)$, since $b < w_l$. Therefore $\Delta_l = 0$.

Step 2: Suppose, there exists b, $b < w_l$ such that $\Delta_h > 0$. In this step and the next we focus on the case $b \geq w_h - w_l$. For the other case, as was shown in the previous step, identical techniques will apply to complete the payoff. The payoff for a high a budget bidder who bids b is given below

$$\begin{aligned}
\Pi_h(b) = {} & (1-\rho)^2\left[F_l(b)^2\mathbb{E}[v-\text{second highest bid}]+(1-F_l(b)^2)(v-w_l)\right] \\
& +2\rho(1-\rho)\Bigg[F_l(b)(F_h(b)-\Delta_h)\mathbb{E}[v-\text{second highest bid}]+F_l(b)\Delta_h\left(\frac{v-b}{2}-\frac{v-w_l}{2}\right) \\
& \qquad + F_l(b)(1-F_h(b))(v-w_l)+(1-F_l(b))(F_h(b)-\Delta_h)\left(\frac{v-w_h}{2}\right) \\
& +(1-F_l(b))\Delta_h\left(\frac{v-w_h}{2}\right)+(1-F_l(b))(1-F_h(b))\Bigg((\Pr h \text{ wins first auction})(v-w_l) \\
& \qquad + (\Pr l \text{ wins first auctions})\left(\frac{v-w_h}{2}\right)\Bigg)\Bigg] \\
& +\rho^2\Bigg[(F_h(b)-\Delta_h)^2\mathbb{E}[v-\text{second highest bid}]+2(F_h(b)-\Delta_h)\Delta_h\left(\frac{v-b}{2}+\frac{v-w_h}{4}\right) \\
& \qquad + \Delta_h^2\left(\frac{v-b}{3}+\frac{2}{3}\left(\frac{v-w_h}{2}\right)\right)+(1-F_h(b)^2)\left(\frac{v-w_h}{2}\right)\Bigg]
\end{aligned}$$

Below we calculate the payoff for a high budget bidder from bidding slightly more than b.

$$\begin{aligned}
\Pi_h(b_+) \approx {} & (1-\rho)^2\left[F_l(b)^2\mathbb{E}[v-\text{second highest bid}]+(1-F_l(b)^2)(v-w_l)\right] \\
& + 2\rho(1-\rho)\Bigg[F_l(b)(F_h(b)-\Delta_h)\mathbb{E}[v-\text{second highest bid}]+F_l(b)\Delta_h(v-b) \\
& \qquad + F_l(b)(1-F_h(b))(v-w_l)+(1-F_l(b))(F_h(b)-\Delta_h)\left(\frac{v-w_h}{2}\right) \\
& \qquad + (1-F_l(b))\Delta_h\left(\frac{v-w_h}{2}\right) \\
& \qquad + (1-F_l(b))(1-F_h(b))\Bigg((\Pr h \text{ wins first auction})(v-w_l) \\
& \qquad + (\Pr l \text{ wins first auctions})\left(\frac{v-w_h}{2}\right)\Bigg)\Bigg] \\
& + \rho^2\Bigg[(F_h(b)-\Delta_h)^2\mathbb{E}[v-\text{second highest bid}]+2(F_h(b)-\Delta_h)\Delta_h(v-b) \\
& \qquad + \Delta_h^2(v-b)+(1-F_h(b)^2)\left(\frac{v-w_h}{2}\right)\Bigg]
\end{aligned}$$

Again, as in the previous step $\Pi_h(b_+) > \Pi_h(b)$ since $b < w_l$. Therefore $\Delta_h = 0$.

Step 3: Suppose $b > w_l$ and $\Delta_h > 0$. Then $F_l(b) = 1$. The difference in payoffs from bidding b and slightly higher is given approximately by

$$\Pi_h(b) - \Pi_h(b_+) \approx 2\rho(1-\rho)\Delta_h \left(\frac{v-w_l}{2} - \frac{v-b}{2}\right) + \rho^2 \left(2(F_h(b)-\Delta_h)\Delta_h \left(\frac{v-w_h}{4} - \frac{v-b}{2}\right) + \frac{2\Delta_h^2}{3}\left(\frac{v-w_h}{2} - (v-b)\right)\right)$$

If bidding b is optimal then the above difference must be weakly positive. That is,

$$\rho\Delta_h \left((1-\rho)(b-w_l) + \rho(2b-w_h-v)\left(\frac{F_h(b)}{2} - \frac{\Delta_h}{6}\right)\right) \geq 0 \tag{12}$$

Now, consider the bidder bidding b_- which is slightly less than b.

$$\begin{aligned}\Pi_h(b_-) \approx\ & (1-\rho)^2 \left[F_l(b)^2\mathbb{E}[v - \text{second highest bid}] + (1-F_l(b)^2)(v-w_l)\right] \\ & + 2\rho(1-\rho)\Big[F_l(b)(F_h(b)-\Delta_h)\mathbb{E}[v - \text{second highest bid}] + F_l(b)\Delta_h(v-w_l) \\ & + F_l(b)(1-F_h(b))(v-w_l) + (1-F_l(b))(F_h(b)-\Delta_h)\left(\frac{v-w_h}{2}\right) \\ & + (1-F_l(b))\Delta_h\left(\frac{v-w_h}{2}\right) \\ & + (1-F_l(b))(1-F_h(b))\Big((\text{Pr } h \text{ wins first auction})(v-w_l) \\ & + (\text{Pr } l \text{ wins first auctions})\left(\frac{v-w_h}{2}\right)\Big)\Big] \\ & + \rho^2\Big[(F_h(b)-\Delta_h)^2\mathbb{E}[v - \text{second highest bid}] + 2(F_h(b)-\Delta_h)\Delta_h\left(\frac{v-w_h}{2}\right) \\ & + \Delta_h^2\left(\frac{v-w_h}{2}\right) + (1-F_h(b)^2)\left(\frac{v-w_h}{2}\right)\Big]\end{aligned}$$

Below, we calculate the approximate difference in payoffs from bidding b and slightly lower.

$$\Pi_h(b) - \Pi_h(b_-) \approx 2\rho(1-\rho)\Delta_h\left(\frac{v-b}{2} - \frac{v-w_l}{2}\right) + \rho^2\left(2(F_h(b)-\Delta_h)\Delta_h\left(\frac{v-b}{2} - \frac{v-w_h}{4}\right) + \Delta_h\left(\frac{v-b}{3} - \frac{v-w_h}{6}\right)\right)$$

If bidding b is optimal then reducing the bid slightly should weakly lower the payoff. That is the above difference must be weakly larger than zero. Therefore,

$$\rho\Delta_h\left((1-\rho)(w_l-b) + \rho(v+w_h-2b)\left(\frac{F_h(b)}{2} - \frac{\Delta_h}{3}\right)\right) \geq 0 \tag{13}$$

Since $b > w_l, b < w_h < v$ and $\Delta_h > 0$, equations (12) and (13) contradict each other. □

Proof of Lemma 3.4: Consider bidder type l. Since $b \in \operatorname{supp} F_l$, there exists a sequence $b_n \to b$, such that $\Pi_l(b_n) = \Pi_l(b_{n+1}) \geq \Pi_l(b')$. That is the payoff is identical at any point in the sequence and its weakly larger that the payoff from bidding b'. To see this, note that for all $\varepsilon > 0$, $F_i(b+\varepsilon) - F_i(b-\varepsilon) > 0$, by definition. Fix ε and let $\varepsilon_n = \frac{\varepsilon}{n}$. For each n we can find $b_n \in B_{e_n}(b)$, where B represents an open ball around b, such that $b_n \in E_l$. Now, we know $\Pi_l(b_n) \geq \Pi_l(b')$ for all n. The payoff function can only 'jump up' or 'drop down' if there is a discontinuity in the payoff function, which itself can only occur if there is a discontinuity in the bid distributions F_l or F_h at b, which can not happen. Hence $\Pi_l(b) = \lim_{n\to\infty} \Pi_l(b_n)$.
□

Proof of Lemma 3.5: Suppose $\hat{b} \in \operatorname{supp} F_i$ for $i = l, h$ and $\hat{b} < w_l$. Then there must exist $b \in (\hat{b} - \varepsilon, \hat{b} + \varepsilon)$, with $\hat{b} + \varepsilon < w_l$, such that $f_l(b) > 0$, or $f_h(b) > 0$, or both, where $f_i(\cdot)$ is the probability density function of the bid distributions. To see this, note the following two observations.

First, note that since F_i's are monotone functions, they must be differentiable almost everywhere.[13] Hence the function f_i exists almost everywhere. Second, due to Lemma 3.2 we know that both bid distributions are continuous at $\hat{b}$. This means that they will also be continuous in a neighborhood $(\hat{b} - \varepsilon, \hat{b} + \varepsilon)$, where $F_i(\hat{b}+\varepsilon) - F_i(\hat{b}-\varepsilon) > 0$, since $\hat{b} \in \operatorname{supp} F_i$. Since f_i's are well defined almost everywhere, if not everywhere, over $(\hat{b} - \varepsilon, \hat{b} + \varepsilon)$, there must exist $b \in (\hat{b} - \varepsilon, \hat{b} + \varepsilon)$, such that both $f_l(b)$ and $f_h(b)$ are well defined and at least one is strictly positive.[14]

Before we state the payoff of the bidder who bids b in the first auction, a technical comment. Suppose $F_i(\cdot)$ is continuous over $[c, d]$. Since F_i is differentiable almost everywhere, any value $\int_c^d h(x)dF_i(x)$ is well defined, where $h(\cdot)$ is some function defined over the support. That

[13]Theorem 5.3 in Royden (see page 100).

[14]If $f_l(b)$ and $f_h(b)$ are well defined but are always zero, then F_i will be flat over $(\hat{b} - \varepsilon, \hat{b} + \varepsilon)$.

is $h(x)dF_i(x)$ is Riemann-integrable over the support. This is so, because for any points of non-differentiability, we can assign $dF_i(x) = 0$ and it will not affect the value of the integral as a set containing only such points is of measure zero.

We state the payoff for a low budget bidder who bids b using (1). Let $\underline{b} = \min\{\inf\{x|x \in \operatorname{supp} F_l\}, \inf\{x|x \in \operatorname{supp} F_h\}\}$. Since $b, \underline{b} < w_l$, F_l and F_h are continuous over $[\underline{b}, b]$, the payoff function is well defined.

$$
\begin{aligned}
\Pi_l^{ll}(b) =&(1-\rho)^2 \left[F_l(b)^2 \int_{\underline{b}}^{b} (v-x) \frac{dF_l(x)^2}{F_l(b)^2} + 2F_l(b)(1-F_l(b)) \left(\frac{v-w_l}{2}\right) \right. \\
&\left. + (1-F_l(b))^2 \left(\frac{v-w_l}{2}\right) \right] \\
\Pi_l^{lh}(b) =&2\rho(1-\rho) \left[F_l(b)F_h(b) \int_{\underline{b}}^{b} (v-x) \frac{d\left(F_l(x)F_h(x)\right)}{F_l(b)F_h(b)} + F_l(b)\left(1-F_h(b)\right) \left(\frac{v-w_l}{2}\right) \right. \\
&\left. + \left(1-F_l(b)\right)\left(1-F_h(b)\right) \int_{b}^{\bar{b}} \frac{F_l(x)-F_l(b)}{1-F_l(b)} \left(\frac{v-w_l}{2}\right) d\frac{F_h(x)-F_h(b)}{1-F_h(b)} \right] \\
\Pi_l^{hh}(b) =&\rho^2 F_h(b)^2 \int_{\underline{b}}^{b} (v-x) \frac{dF_h(x)^2}{F_h(b)^2}
\end{aligned}
$$

Consider $\Pi_l^{ll}(b)$. A bidder faces two low budget bidders with probability $(1-\rho)^2$. When facing two low budget bidders who follow the strategy F_l in the first auction and bid their remaining budgets in the second, a bidder who bids b in the first auction can have three outcomes. First, she may beat both the bids of other bidders with probability $F_l(b)^2$. In this case her payoff will be her valuation minus the expected highest bid of the other two bidders, conditional on both bids being less than b. In the second scenario she may defeat one of the bidders but lose to the other which happens with probability $F_l(b)(1-F_l(b))$. In this case the bidder will lose the first auction and then bid w_l in the second auction, where she will tie with the other loser from the first auction, who also bids w_l. Finally, she may lose to both the other bidders in the first auctions which occurs with probability $(1-F_l(b))^2$. In this case, again there will be two bidders in the final round with their budgets intact and there will be tie. The other parts of the payoff function are defined similarly. Adding the three parts as in (1) and simplifying,

$$\begin{aligned}
\Pi_l(b) = (1-\rho)^2 &\left[\int_{\underline{b}}^{b} (v-x)2F_l(x)f_l(x)dx + \left(1-F_l(b)^2\right)\left(\frac{v-w_l}{2}\right)\right] \\
&+ 2\rho(1-\rho)\left[\int_{\underline{b}}^{b}(v-x)F_h(x)f_l(x)dx + \int_{\underline{b}}^{b}(v-x)F_l(x)f_h(x)dx \right. \\
&\left. + \left(\frac{v-w_l}{2}\right)\int_b^{\bar{b}} F_l(x)f_h(x)dx\right] + \rho^2\int_{\underline{b}}^{b}(v-x)2F_h(x)f_h(x)dx
\end{aligned} \tag{14}$$

Since $f_l(b)$ and $f_h(b)$ exist, the slope of the payoff function is given by

$$\begin{aligned}
\frac{d\Pi_l(b)}{db} &= (1-\rho)^2\left[2f_l(b)F_l(b)(v-b) - F_l(b)f_l(b)(v-w_l)\right] \\
&\quad + 2\rho(1-\rho)\left[(v-b)F_h(b)f_l(b) + (v-b)F_l(b)f_h(b) - \left(\frac{v-w_l}{2}\right)F_l(b)f_h(b)\right] \\
&\quad + 2\rho^2(v-b)F_h(b)f_h(b) \\
&= \Big((1-\rho)(v-2b+w_l)F_l(b) + 2\rho(v-b)F_h(b)\Big)\left((1-\rho)f_l(b)+\rho f_h(b)\right) > 0
\end{aligned} \tag{15}$$

The final inequality follows from $b < w_l < v$ and, $f_l(b) > 0$ or $f_h(b) > 0$ or both. Since $b \in \operatorname{supp} F_l$, Lemma 3.4 implies, $\Pi_l(b) \geq \Pi_l(b')$ for all b'. However, the final inequality implies that bidding a little more than b increases the low budget bidder's payoff, which is a contradiction. □

Proof of Lemma 3.6: Suppose for some $b < w_l$, $F_l(b) > 0$. Then, let $\underline{b}_l = \inf\{x \mid F_l(x+\varepsilon) > 0;\text{ for all } \varepsilon > 0\}$, and $\underline{b}_l < w_l$. Note that $\underline{b}_l \in \operatorname{supp} F_l$, from definition of the latter and the fact that it is a closed set. We show that there exists b' such that $\Pi_l(b') > \Pi_l(b)$, contradicting Lemma 3.4

It must be the case that $F_l(\underline{b}_l) = 0$, due to the definition of $\underline{b}_l$ and the continuity of $F_l(\cdot)$ at $\underline{b}_l$. Then the payoff to a low budget bidder from bidding $\underline{b}_l$ is given in equation (16), where $\underline{b} = \min\{\inf\{x|x \in \operatorname{supp} F_l\}, \inf\{x|x\in \operatorname{supp} F_h\}\}$.[15]

[15]In writing the payoff function we are assuming $\underline{b}_l > w_h - w_l$. The case $\underline{b}_l \leq w_h - w_l$ can be analyzed by replacing the 'probability h wins first auction' with 'probability h wins first auction at a price greater than

$$\Pi_l(\underline{b}_l) = (1-\rho)^2 \left(\frac{v-w_l}{2}\right) + 2\rho(1-\rho)\left[(\text{Pr h wins first auction})\left(\frac{v-w_l}{2}\right)\right] + \rho^2 F_h(\underline{b}_l)^2 \int_{\underline{b}}^{\underline{b}_l} (v-x)\frac{dF_h(x)^2}{F_h(\underline{b}_l)^2}. \tag{16}$$

Now let us suppose the bidder bids $\hat{b}$ in the first auction, where $\hat{b} > \underline{b}_l$ but only slightly. Since $F_l(\underline{b}_l + \varepsilon) > 0$ for all $\varepsilon > 0$, $F_l(\cdot)$ must be strictly increasing just to the right of $\underline{b}_l$. Without loss of generality, $F_l(\cdot)$ is strictly increasing over $[\underline{b}_l, \hat{b}]$. This implies that if $b \in [\underline{b}_l, \hat{b}]$, then $b \in \operatorname{supp} F_l$. Then, Lemma 3.5 implies that $F_h(\cdot)$ must be flat over $[\underline{b}_l, \hat{b}]$. This implies that the second and third parts of the above payoff do not change if the the bidder bids $\hat{b}$ in the first auction. So the only change will be in the first part, that is, in the case the bidders is competing against two other low budget bidders.

(17)

$$\begin{aligned}
\Pi(\hat{b}) = (1-\rho)^2 \Bigg[& F_l(\hat{b})^2 \int_{\underline{b}_l}^{\hat{b}} (v-x)\frac{dF_l(x)^2}{F_l(\hat{b})^2} + 2F_l(\hat{b})(1-F_l(\hat{b}))\left(\frac{v-w_l}{2}\right) \\
& + \left(1-F_l(\hat{b})\right)^2 \left(\frac{v-w_l}{2}\right)\Bigg] \\
& + 2\rho(1-\rho)\left[(\text{Pr h wins first auction})\left(\frac{v-w_l}{2}\right)\right] + \rho^2 F_h(\underline{b}_l)^2 \int_{\underline{b}}^{\underline{b}_l} (v-x)\frac{dF_h(x)^2}{F_h(\underline{b}_l)^2} \\
= (1-\rho)^2 & \left[\int_{\underline{b}_l}^{\hat{b}} (v-x)dF_l(x)^2 + (1-F_l(\hat{b})^2)\left(\frac{v-w_l}{2}\right)\right] \\
& + 2\rho(1-\rho)\left[(\text{Pr h wins first auction})\left(\frac{v-w_l}{2}\right)\right] + \rho^2 F_h(\underline{b}_l)^2 \int_{\underline{b}}^{\underline{b}_l} (v-x)\frac{dF_h(x)^2}{F_h(\underline{b}_l)^2}.
\end{aligned}$$

$w_h - w_l$. Again, we are following the convention that a bidder can only win auction, in this section, leaving the other case for section 5.3.

Subtracting (16) from (17) we get

$$
\begin{aligned}
\Pi(\hat{b}) - \Pi(\underline{b}_l) &= (1-\rho)^2 \left[\int_{\underline{b}_l}^{\hat{b}} (v-x) dF_l(x)^2 + (1 - F_l(\hat{b})^2) \left(\frac{v - w_l}{2} \right) - \frac{v - w_l}{2} \right] \\
&= (1-\rho)^2 \left[F_l(x)^2 (v-x) \Big|_{\underline{b}_l}^{\hat{b}} - \int_{\underline{b}_l}^{\hat{b}} F_l(x)^2 (-dx) - F_l(\hat{b})^2 \left(\frac{v - w_l}{2} \right) \right] \\
&= (1-\rho)^2 \left[F_l(\hat{b})^2 \left(\frac{v - 2\hat{b} + w_l}{2} \right) + \int_{\underline{b}_l}^{\hat{b}} F_l(x)^2 (dx) \right] > 0.
\end{aligned} \tag{18}
$$

The second equality follows from integration by parts. The third follows from the fact that $F_l(\underline{b}_l) = 0$. Finally the difference is positive due to $\hat{b} < w_l < v$. Hence $\underline{b}_l \notin \operatorname{supp} F_l$, which is a contradiction. □

Proof of Lemma 3.7: Suppose not. To begin with, suppose a high budget bidder bids w_l in the first auction. Proving a contradiction in this case is similar to the proof of Lemma 3.6, therefore we state a sketch of the proof. We know, through Lemma 3.6, that if the high budget bids w_l in the first auction then she would tie with the low budget bidders for sure. In this case she would win the first auction only half the time. If she wins she would pay w_l but if she loses he would pay either w_l or, if there is another high budget bidder who loses the first auction, w_h in the second round. If instead the high budget bidder bids slightly more than w_l then she would surely defeat all the low budget bidders with probability 1 and only pay w_l (or slightly higher). Since $w_l < w_h$, the bidder is better off (by not having to pay w_h in the second round with a lower probability- that probability is shifted to the first round, where he only pays w_l)

Now, suppose $b < w_l$ and $b \in \operatorname{supp} F_h$. Then we can find $\underline{b}_h = \inf\{x \mid F_h(x+\varepsilon) > 0; \text{ for all } \varepsilon > 0\}$. Just as in the proof of Lemma 3.6 it must be the case that, $\underline{b}_h \in \operatorname{supp} F_h$. Applying an identical reasoning as in the proof of Lemma 3.6, we can show that the high budget bidder gets a higher payoff from bidding slightly more than $\underline{b}_h$, a contradiction to Lemma 3.4. □

Proof of Lemma 3.8: Suppose, $w_h > \frac{6(1-\rho)}{6-5\rho} w_l + \frac{\rho}{6-5\rho} v$ but in equilibrium $F_h(b) = 0$, for all $b < w_h$. Then the high budget bidder's payoff from bidding w_h is

(19)
$$\begin{aligned}\Pi_h(w_h) &= (1-\rho)^2(v-w_l) + 2\rho(1-\rho)\left(\frac{v-w_h}{2} + \frac{v-w_l}{2}\right) + \rho^2\left(\frac{v-w_h}{3} + \frac{2}{3}\left(\frac{v-w_h}{2}\right)\right)\\ &= (1-\rho)^2(v-w_l) + 2\rho(1-\rho)\left(\frac{2v-w_h-w_l}{2}\right) + \rho^2\left(\frac{2(v-w_h)}{3}\right).\end{aligned}$$

Suppose this high budget bidder bids slightly less than w_h, then her payoff will be

$$\Pi(w_{h-}) = (1-\rho)^2(v-w_l) + 2\rho(1-\rho)(v-w_l) + \rho^2\left(\frac{v-w_h}{2}\right). \tag{20}$$

Subtracting equation (19) from (20),

$$\begin{aligned}\Pi(w_{h-}) - \Pi_h(w_h) &= 2\rho(1-\rho)\left(\frac{w_h-w_l}{2}\right) - \rho^2\left(\frac{v-w_h}{6}\right)\\ &= \rho\left(w_h\left(\frac{6-5\rho}{6}\right) - \frac{\rho}{6}v + (1-\rho)w_l\right) > 0,\end{aligned}$$

where the final inequality follows from the condition on w_h. Therefore, it can not be an equilibrium for the high budget bidders to always bid their budgets in the first round, and hence, $F_h(b) > 0$ for some $b < w_h$. This proves sufficiency.

For necessity, suppose that $w_h \leq \frac{6(1-\rho)}{6-5\rho} w_l + \frac{\rho}{6-5\rho} v$ but the high type bidders bid their budgets w_h with probability $\Delta_h < 1$. If $\Delta_h = 0$, then by Lemma 3.2 the bid distribution $F_h(\cdot)$ is continuous. Also, $F_h(\cdot)$ is differentiable almost everywhere. Let $f_h(\cdot)$ denote the corresponding density function. Let $\bar{b}_h \equiv \sup\{\operatorname{supp} F_h\}$ and $\underline{b}_h \equiv \inf\{\operatorname{supp} F_h\}$ The payoff to a high budget bidder from bidding $b \in \operatorname{supp} F_h$ is given by:

$$\begin{aligned}
\Pi_h(b) &= (1-\rho)^2(v-w_l) + 2\rho(1-\rho)\left[F_h(b)\int_{\underline{b}_l}^{b}(v-x)\frac{dF_h(x)}{F_h(b)} + (1-F_h(b))(v-w_l)\right] \\
&\quad + \rho^2\left[F_h(b)^2\int_{\underline{b}_l}^{b}(v-x)\frac{dF_h(x)^2}{F_h(b)^2} + (1-F_h(b)^2)\left(\frac{v-w_h}{2}\right)\right] \\
&= (1-\rho)^2(v-w_l) + 2\rho(1-\rho)\left[\int_{\underline{b}_l}^{b}(v-x)f_h(x)dx + (1-F_h(b))(v-w_l)\right] \\
&\quad + \rho^2\left[\int_{\underline{b}_l}^{b}(v-x)2F_h(x)f_h(x)dx + (1-F_h(b)^2)\left(\frac{v-w_h}{2}\right)\right].
\end{aligned}$$

Without loss of generality, suppose b is close to $\bar{b}_h$. Since $F_h(\cdot)$ is differentiable almost everywhere, again, without loss of generality $f_h(b)$ exists. Then taking the derivative of the payoff, we get

$$\frac{d\Pi_h(b)}{db} = 2\rho(1-\rho)f_h(b)(w_l-b) + \rho^2F_h(b)f_h(b)(v-2b+w_h) \leq 0,$$

where the inequality follows $b \in \operatorname{supp} F_h$.[16] Since b is close to $\bar{b}_h$ and $F_h(\bar{b}_h-\varepsilon) - F_h(\bar{b}_h) > 0$ for all $\varepsilon > 0$, it must be the case that $f_h(b) > 0$, generically. Therefore the above equation implies a condition on b:

$$b \geq \frac{2(1-\rho)}{2(1-\rho)+2\rho F_h(b)}w_l + \frac{\rho F_h(b)}{2(1-\rho)+2\rho F_h(b)}(v+w_h).$$

As $b \to \bar{b}_h$, $\lim_{b\to\bar{b}_h} F_h(b) = F_h(\bar{b}_h) = 1$ due to continuity of $F_h(\cdot)$. Evaluating the above condition as $b \to \bar{b}_h$, we get a condition on $\bar{b}_h$. And finally, since $\bar{b}_h \leq w_h$, it follows that

$$w_h \geq \frac{2(1-\rho)}{2-\rho}w_l + \frac{\rho}{2-\rho}v > \frac{6(1-\rho)}{6-5\rho}w_l + \frac{\rho}{6-5\rho}v,$$

which contradicts the assumption that $w_h \leq \frac{6(1-\rho)}{6-5\rho}w_l + \frac{\rho}{6-5\rho}v$.

[16] The slope of the payoff function at b must be weakly negative, due to Lemma 3.4.

If $\Delta_h \in (0,1)$, then by Lemma 3.2 the bid distribution $F_h(\cdot)$ is continuous except at w_h. Let $\hat{b} = \sup\{\text{supp}\, F_h \setminus \{w_h\}\}$. The payoff from bidding $\hat{b}$ and w_h are

$$\begin{aligned} \Pi_h(w_h) &= (1-\rho)^2(v-w_l) + 2\rho(1-\rho)\left[\int_{w_l+}^{\hat{b}} (v-x)dF_h(x) + \Delta_h\left(\frac{v-w_h}{2} + \frac{v-w_l}{2}\right)\right] \\ &\quad + \rho^2\left[\int_{w_l+}^{\hat{b}} (v-x)dF_h(x)^2 + 2\Delta_h(1-\Delta_h)\left(\frac{3}{4}(v-w_h)\right) + \Delta_h^2\left(\frac{2}{3}(v-w_h)\right)\right], \end{aligned} \tag{21}$$

and

$$\begin{aligned} \Pi_h(\hat{b}) &= (1-\rho)^2(v-w_l) + 2\rho(1-\rho)\left[\int_{w_l+}^{\hat{b}} (v-x)dF_h(x) + \Delta_h(v-w_l)\right] \\ &\quad + \rho^2\left[\int_{w_l+}^{\hat{b}} (v-x)dF_h(x)^2 + 2\Delta_h(1-\Delta_h)\left(\frac{v-w_h}{2}\right) + \Delta_h^2\left(\frac{v-w_h}{2}\right)\right], \end{aligned} \tag{22}$$

respectively.

Note that $w_h \in \text{supp}\, F_h$. Therefore, $\Pi_h(w_h) = \Pi_h(\hat{b})$, due to Lemma 3.4. This implies

$$\Delta_h = \frac{3}{2}\left(1 - \frac{2(1-\rho)(w_h - w_l)}{\rho(v-w_h)}\right). \tag{23}$$

Since $\Delta_h < 1$, it must be the case that $(6-5\rho)w_h > 6(1-\rho)w_l + \rho v$ which contradicts the assumption that $w_h \leq \frac{6(1-\rho)}{6-5\rho}w_l + \frac{\rho}{6-5\rho}v$. $\square$

Proof of Lemma 3.9: If $w_h \geq \frac{2(1-\rho)}{2-\rho}w_l + \frac{\rho}{2-\rho}v$, then $w_h > \frac{6(1-\rho)}{6-5\rho}w_l + \frac{\rho}{6-5\rho}v$. Therefore the bidder must be randomizing in equilibrium. Let us suppose the bidder randomizes over bids contained in $[\underline{b}_h, \bar{b}_h]$, where $\underline{b}_h = \inf\{\text{supp}\, F_h\}$ and $\bar{b}_h = \sup\{\text{supp}\, F_h \setminus \{w_h\}\}$.[17]

Due to Lemma 3.2 we know that $F_h(\cdot)$ is continuous over $[\underline{b}_h, \bar{b}_h)$. Since F_h is monotone, we know that it is differentiable almost everywhere. Hence the density $f_h(\cdot)$ exists almost everywhere in $(\underline{b}_h, \bar{b}_h)$. Hence we can write down the payoff function of a high budget bidder

[17]Notice that $\bar{b}_h$ can still equal w_h, if the bidder randomizes continuously over $[\underline{b}_h, w_h]$. Also, at this stage we are not assuming that she does not bid w_h with positive probability in equilibrium.

who bids $b \in \operatorname{supp} F_h$ as follows.

$$\begin{aligned}\Pi_h(b) &= (1-\rho)^2(v-w_l) + 2\rho(1-\rho)\left[F_h(b)\int_{\underline{b}_h}^{b}(v-x)\frac{dF_h(x)}{F_h(b)} + (1-F_h(b))(v-w_l)\right] \\ &\quad + \rho^2\left[F_h(b)^2\int_{\underline{b}_h}^{b}(v-x)\frac{dF_h(x)^2}{F_h(b)^2} + (1-F_h(b)^2)\left(\frac{v-w_h}{2}\right)\right] \\ &= (1-\rho)^2(v-w_l) + 2\rho(1-\rho)\left[\int_{\underline{b}_h}^{b}(v-x)f_h(x)dx + (1-F_h(b))(v-w_l)\right] \\ &\quad + \rho^2\left[\int_{\underline{b}_h}^{b}(v-x)2F_h(x)f_h(x)dx + (1-F_h(b)^2)\left(\frac{v-w_h}{2}\right)\right].\end{aligned}$$

Without loss of generality we can assume that $f_h(b)$ exists, since the density is surely defined almost everywhere, if not everywhere. We can also assume $f_h(b) > 0$. If we can not find such a bid in $(\underline{b}_h, \bar{b}_h)$, that would imply $F_h(b) = 0$ for all $b < w_h$ and hence the bidder bids w_h for sure, a contradiction.

Taking the derivative of the payoff function we can measure the change in the payoff to the bidder from bidding a little more than b.

$$\begin{aligned}\frac{d\Pi_h}{db} &= 2\rho(1-\rho)\left[(v-b)f_h(b) - f_h(b)(v-w_l)\right] + \rho^2\left[2(v-b)F_h(b)f_h(b) - F_h(b)f_h(b)(v-w_h)\right] \\ &= 2\rho(1-\rho)f_h(b)(w_l-b) + \rho^2 F_h(b)f_h(b)(v-2b+w_h).\end{aligned}$$

This derivative must equal zero. If not, it would imply that there exists a profitable deviation for the bidder in which case $b \notin \operatorname{supp} F_h$, a contradiction. Therefore equating to zero and using the fact that $f_h(b) > 0$ and $\rho \in (0,1)$, we get a solution for F_h:

$$F_h(b) = \frac{2(1-\rho)(b-w_l)}{\rho(v+w_h-2b)}. \tag{24}$$

Notice that $F_h(w_l) = 0$ and $F(b) > 0$ for $b > w_l$. From Lemma 3.7 we also know that a high budget bidder will never bid w_l in equilibrium. So, we assume that high bidder's lowest possible bid in the first round is w_{l+}, such that $w_{l+} > w_l$ but $w_{l+} \in B_\varepsilon(w_l)$ for all $\varepsilon > 0$. So, $\underline{b}_h = w_{l+}$.[18]

[18] Adding the possible bid w_{l+} guarantees equilibrium existence. This idea has been widely used in the auction literature. See Engelbrecht-Wiggans and Weber (1983), Hörner and Jamison (2008), and Bergemann and Hörner (2010).

Note that $F'_h(b) > 0$. Therefore for $b < b'$, $F_h(b) < F_h(b')$. Hence the bid distribution can not have any 'flat' portions. That is the bid distribution is strictly increasing and supp F_h is a convex set. Given the functional form of F_h, it must be the case that for all $b \in [\underline{b}_h, \bar{b}_h]$, $f_h(b)$ is well defined and is given by the $F'_h(b)$.

Now, suppose that $F(\bar{b}_h) = 1$. Using equation (24), we can find the value of $\bar{b}_h$ to be $(1-\rho)w_l + \frac{\rho(v+w_h)}{2}$, as stated in proposition 3.1. If $\bar{b}_h \leq w_h$, then it is possible for this equilibrium to exist. It is straightforward to check that this condition implies $w_h \geq \frac{2(1-\rho)}{2-\rho}w_l + \frac{\rho}{2-\rho}v$. $\square$

Proof of Lemma 3.10: In this case the equilibrium can not be degenerate nor can it include only continuous mixing by the high budget bidder. Therefore, the only possibility remaining is that the high budget bidder will mix continuously over some interval but also bid w_h in the first auction with positive probability.[19] Suppose the bidder bids w_h in the first auction with probability Δ_h.

Suppose the bidder randomizes over bids contained in $[\underline{b}_h, \hat{b}]$, where $\underline{b}_h = \inf\{\text{supp}\, F_h\}$ and $\hat{b} = \sup\{\text{supp}\, F_h/\{w_h\}\}$. Then using an identical method of constructing an equilibrium bid distribution as was used in the proof of Lemma 3.9, we can show that the bid distribution will be as in equation (24) over $[\underline{b}_h, \hat{b}]$, with $\underline{b}_h = w_{l+}$.

It must be the case that $\hat{b} < w_h$. To prove this, let us consider the case $\hat{b} = w_h$. Suppose $\Delta_h = 0$. Then the bidder is only mixing (continuously) over a set of bids, which can not be the case.

Now, suppose $\Delta_h > 0$. We can calculate this probability mass since we know that $\Delta_h = 1 - \lim_{b \to w_h} F_h(b)$, that is,

$$\Delta_h = 1 - \frac{2(1-\rho)(w_h - w_l)}{\rho(v - w_h)}. \tag{25}$$

[19]Due to Lemma 3.2 the bid distribution is continuous everywhere but, possibly, w_h.

The payoff to a high budget bidder from bidding w_h is given by

(26)

$$\Pi_h(w_h) = (1-\rho)^2(v-w_l) + 2\rho(1-\rho)\left[(1-\Delta_h)\int_{w_{l+}}^{w_h}(v-x)\frac{dF_h(x)}{1-\Delta_h} + \Delta_h\left(\frac{v-w_h}{2}+\frac{v-w_l}{2}\right)\right]$$
$$+\rho^2\left[(1-\Delta_h)^2\int_{w_{l+}}^{w_h}(v-x)\frac{dF_h(x)^2}{(1-\Delta_h)^2} + 2\Delta_h(1-\Delta_h)\left(\frac{v-w_h}{2}+\frac{v-w_h}{4}\right)\right.$$
$$\left. + \Delta_h^2\left(\frac{v-w_h}{3}+\frac{2}{3}\left(\frac{v-w_h}{2}\right)\right)\right]$$
$$= (1-\rho)^2(v-w_l) + 2\rho(1-\rho)\left[\int_{w_{l+}}^{w_h}(v-x)dF_h(x) + \Delta_h\left(\frac{v-w_h}{2}+\frac{v-w_l}{2}\right)\right]$$
$$+\rho^2\left[\int_{w_{l+}}^{w_h}(v-x)dF_h(x)^2 + 2\Delta_h(1-\Delta_h)\left(\frac{3}{4}(v-w_h)\right) + \Delta_h^2\left(\frac{2}{3}(v-w_h)\right)\right].$$

Now consider the bidder bidding w_{h-}, which is a bid slightly less than w_h. Since $\hat{b} = \sup\{\operatorname{supp} F_h \setminus \{w_h\}\}$ by definition and $\hat{b} = w_h$ by assumption in this part of the proof, we can always find $w_{h-} \in \operatorname{supp} F_h$, as close to w_h as we like.

(27)

$$\Pi_h(w_{h-}) = (1-\rho)^2(v-w_l)$$
$$+2\rho(1-\rho)\left[F_h(w_{h-})\int_{w_{l+}}^{w_{h-}}(v-x)\frac{dF_h(x)}{F_h(w_{h-})} + (1-F_h(w_{h-}))(v-w_l)\right]$$
$$+\rho^2\left[(F_h(w_{h-}))^2\int_{w_{l+}}^{w_{h-}}(v-x)\frac{dF_h(x)^2}{(F_h(w_{h-}))^2} + 2F_h(w_{h-})(1-F_h(w_{h-}))\left(\frac{v-w_h}{2}\right)\right.$$
$$\left. + (1-F_h(w_{h-}))^2\left(\frac{v-w_h}{2}\right)\right],$$

where we explicitly use the fact that F_h must be continuous and differentiable at w_{h-}. Simplifying and substituting $F_h(w_{h-}) \approx 1-\Delta_h$, since F_h is continuous at w_{h-} and w_{h-} is arbitrarily close to w_h,

$$\Pi_h(w_{h-}) = (1-\rho)^2(v-w_l) + 2\rho(1-\rho)\left[\int_{w_{l+}}^{w_h}(v-x)dF_h(x) + \Delta_h\,(v-w_l)\right] \tag{28}$$
$$+ \rho^2\left[\int_{w_{l+}}^{w_h}(v-x)dF_h(x)^2 + 2\Delta_h(1-\Delta_h)\left(\frac{v-w_h}{2}\right) + \Delta_h^2\left(\frac{v-w_h}{2}\right)\right].$$

Subtracting $\Pi_h(w_h)$ from $\Pi_h(w_{h-})$ we get,

$$\begin{aligned}\Pi_h(w_{h-}) - \Pi_h(w_h) &= 2\rho(1-\rho)\Delta_h\left(\frac{v-w_l}{2} - \frac{v-w_h}{2}\right)\\ &\quad + \rho^2\left(-2\Delta_h(1-\Delta_h)\left(\frac{v-w_h}{4}\right) - \Delta_h^2\left(\frac{v-w_h}{6}\right)\right)\\ &= 2\rho(1-\rho)\Delta_h\left(\frac{w_h-w_l}{2}\right) - \rho^2\Delta_h\left((v-w_h)\left(\frac{1}{2}-\frac{\Delta_h}{3}\right)\right)\\ &= \frac{\rho\Delta_h(2-\rho)}{6}\left(w_h - \left(\frac{2(1-\rho)}{2-\rho}w_l + \frac{\rho}{2-\rho}v\right)\right),\end{aligned} \tag{29}$$

where the final equality follows from substituting the value of Δ_h from equation (25). Clearly, $\Pi_h(w_{h-}) - \Pi_h(w_h) < 0$ since $w_h < \frac{2(1-\rho)}{2-\rho}w_l + \frac{\rho}{2-\rho}v$. Therefore, $w_{h-} \notin \operatorname{supp} F_h$, a contradiction. Therefore, $\hat{b} < w_h$.

In order to complete the proof we need to find the value of $\hat{b}$. Before doing that let us compute the probability mass at w_h, since we will use it to calculate $\hat{b}$. It must be the case that the bidder is indifferent between bidding w_h and bidding $\hat{b}$. Equation (26) gives us the payoff to the bidder from bidding w_h, when there is a probability mass at w_h. Similarly, replacing w_{h-} with $\hat{b}$ in equation (27) we get the payoff for a high budget bidder bidding $\hat{b}$. Equating the two, we get the probability mass Δ_h as in (23).[20]

Now, $F_h(\hat{b}) = 1 - \Delta_h$. That is

$$\frac{2(1-\rho)(\hat{b}-w_l)}{\rho(v+w_h-2\hat{b})} = 1 - \Delta_h. \tag{30}$$

[20]Note that if $w_h \geq \frac{2(1-\rho)w_l}{2-\rho} + \frac{\rho}{2-\rho}v$, then $\Delta_h \leq 0$.

Solving for $\hat{b}$, we get

$$\hat{b} = \frac{2(1-\rho)w_l + \rho(1-\Delta_h)(v+w_h)}{2\left((1-\rho)+\rho(1-\Delta_h)\right)} \tag{31}$$

By construction, all bids $b \in [w_{l+}, \hat{b}]$, give the same payoff to the bidder, which is the same as the payoff the bidder gets from bidding w_h. □

Proof of Lemma 3.11: We first focus on a low budget bidder. Suppose the other bidders are bidding according to $F_h(\cdot)$ and $F_l(\cdot)$ as given in proposition 3.1. Then a low budget bidder can choose to bid w_l for sure, as the other bidders, or bid strictly below w_l with positive probability. We show that bidding less than w_l lowers the bidder's payoff. First, we write the payoff from bidding w_l. When writing the payoff function, we omit the part of the payoff which occurs when the bidder competes against two high budget bidder. The reason for this is that If the low budget bidders competes against two high budget bidders, who are playing according to F_h, then his payoff will be zero.

$$\begin{aligned}\Pi_l(w_l) &= (1-\rho)^2\left(\frac{v-w_l}{3}+\frac{2}{3}\left(\frac{v-w_l}{2}\right)\right)+2\rho(1-\rho)\left(\frac{v-w_l}{2}\right)\\ &= (1-\rho)^2\left(\frac{2}{3}(v-w_l)\right)+2\rho(1-\rho)\left(\frac{v-w_l}{2}\right).\end{aligned} \tag{32}$$

Now suppose the bidder bids $b < w_l$. Her payoff in this case will be,

$$\Pi_l(b) = (1-\rho)^2\left(\frac{v-w_l}{2}\right)+2\rho(1-\rho)\left(\frac{v-w_l}{2}\right). \tag{33}$$

Clearly, $\Pi_l(w_l) > \Pi_l(b)$. Therefore, bidding w_l is a best response for ta low budget bidder if the other are playing according to F_l and F_h.

For the high budget bidder, the equilibrium strategies depend on the parameters. Therefore, we focus on each case separately.

Case (i) $w_h \leq \frac{6(1-\rho)}{6-5\rho}w_l + \frac{\rho}{6-5\rho}v$

In this case both the low and high budget bidders are bidding their respective budgets. From Lemma 3.7 we know that a high budget bidder will never bid below w_l in equilibrium.

Therefore, in this case, we only need to consider whether a high budget bidder would put positive probability on the interval (w_l, w_h). If this happens then there must exist $b \in (w_l, w_h)$ such that $\Pi_h(b) \geq \Pi_h(b')$ for all b'.

To see that this cannot happen, consider the proof of Lemma 3.8. Equation (19) gives us the payoff of the high budget bidder if he were to bid w_h in the first auction when the other bidders are bidding according to F_h and F_l. Similarly, replacing w_{h-} with b in equation (20), we get the payoff to a high budget bidder from bidding $b < w_h$, when the others are bidding according to the equilibrium distributions. As in the proof of Lemma 3.8, if $w_h \leq \frac{6(1-\rho)}{6-5\rho} w_l + \frac{\rho}{6-5\rho} v$, then $\Pi_h(w_h) \geq \Pi_h(b)$, with the inequality being strict if the first inequality is strict.

Case (ii) $\frac{6(1-\rho)}{6-5\rho} w_l + \frac{\rho}{6-5\rho} v < w_h < \frac{2(1-\rho)w_l}{2-\rho} + \frac{\rho}{2-\rho} v$

By construction $\Pi_h(b) = \Pi_h(b')$ for all $b, b' \in [w_{l+}, \hat{b}] \cup \{w_h\}$. Therefore, we only need to prove that $\Pi_h(b'') \leq \Pi_h(\hat{b})$ for any $b'' \in (\hat{b}, w_h)$. Note that b'' are the only 'other' bids we need to consider since Lemma 3.7 shows that the high budget bidder will never bid below w_l.

Note that $F_h(b'') = F_h(\hat{b})$, since $F_h(\cdot)$ is flat over $(\hat{b}, w_h)$. Therefore, it must be the case that $\Pi_h(b'') = \Pi_h(\hat{b})$ for all $b'' \in (\hat{b}, w_h)$ as bidding higher than $\hat{b}$ does not affect the payment of a bidder in the case he wins the first round.

Case (iii) $w_h \geq \frac{2(1-\rho)w_l}{2-\rho} + \frac{\rho}{2-\rho} v$

Showing that F_h, F_l form an equilibrium in this case is very similar to the previous case. Bidding more that $\bar{b}$ does not change the probability of winning (which will be one if he bidder bids $\bar{b}$) and hence keeps the payoff the same. And by construction bidding $b \in [w_{l+}, \bar{b})$ gives the same payoff. Therefore, randomizing over $[w_{l+}, \bar{b}]$, using the distribution function F_h is an equilibrium. □

Appendix B

Calculation of $\mathbb{E}(b|w_h)$

We first calculate the first part of the expected bid in equation (2).

$$\begin{aligned}\int_{w_l}^{\hat{b}} x dF_h(x) &= \int_{w_l}^{\hat{b}} x \left(\frac{2(1-\rho)}{\rho}\right)\left(\frac{v+w_h-2w_l}{(v+w_h-2x)^2}\right) dx \\ &= \left(\frac{2(1-\rho)}{\rho}\right)(v+w_h-2w_l)\int_{w_l}^{\hat{b}} \frac{x}{(v+w_h-2x)^2} dx \\ &= \left(\frac{2(1-\rho)}{\rho}\right)(v+w_h-2w_l)\left(\frac{\ln(v+w_h-2x)}{4}+\frac{v+w_h}{4(v+w_h-2x)}\right)\Bigg|_{w_l}^{\hat{b}}.\end{aligned}$$

In evaluating the integral we use the fact that the constant of integration will be zero. Substituting the limits,

$$\begin{aligned}\int_{w_l}^{\hat{b}} x dF_h(x) = \left(\frac{1-\rho}{2\rho}\right)(v+w_h-2w_l)&\left(\ln\left(\frac{v+w_h-2\hat{b}}{v+w_h-2w_l}\right)\right. \\ &\left. + (v+w_h)\left(\frac{2\hat{b}-2w_l}{(v+w_h-2\hat{b})(v+w_h-2w_l)}\right)\right).\end{aligned}$$

Substituting the value of $\hat{b}$ in the above equation we get

$$\int_{w_l}^{\hat{b}} x dF_h(x) = \left(\frac{1-\rho}{2\rho}\right)(v+w_h-2w_l)\ln\left(\frac{1-\rho}{1-\rho+\rho(1-\Delta_h)}\right)+\left(\frac{v+w_h}{2}\right)(1-\Delta_h).$$

Therefore the value of the expected bid is

$$\mathbb{E}(b|w_h) = \left(\frac{1-\rho}{2\rho}\right)(v+w_h-2w_l)\ln\left(\frac{1-\rho}{1-\rho+\rho(1-\Delta_h)}\right)+\frac{v}{2}(1-\Delta_h)+\frac{w_h}{2}(1+\Delta_h).$$

Sketches of Proofs for the case $\mathbf{w_h > 2w_l}$

Lemma 3.2: Continuity of $F_i(\cdot)$ almost everywhere.

In the proof of Lemma 3.2, in step 1, we established that $F_l(b)$ is continuous at all $b < w_l$. In this step we covered the case $b < w_h - w_l$, which is the only case applicable here, since $w_h > 2w_l$ implies lower budget bidder has to bid below $w_h - w_l$. There are two more cases to be considered.

- $b < w_h - w_l$ and $\Delta_h > 0$. Proof is similar to step 2, with changes in the payoff function. The proof will show $\Delta_h = 0$.

- $b = w_h - w_l$ and $\Delta_h > 0$. Proof is similar to step 3, with changes in the payoff function. Again the proof will show $\Delta_h = 0$.

Lemma 3.4: The proof remains the same as the original proof.

Lemma 3.5: We will show the proof for this lemma. Consider the payoff to a low budget bidder from bidding b.

$$\Pi_l(b) = (1-\rho)^2 \left[F_l(b)^2 \int_{\underline{b}}^{b} (v-x) \frac{dF_l(x)^2}{F_l(b)^2} + 2F_l(b)(1-F_l(b)) \left(\frac{v-w_l}{2} \right) + (1-F_l(b))^2 \left(\frac{v-w_l}{2} \right) \right] + 2\rho(1-\rho)F_l(b)F_h(b) \int_{\underline{b}}^{b} (v-x) \frac{d\left(F_l(x)F_h(x)\right)}{F_l(b)F_h(b)} + \rho^2 F_h(b)^2 \int_{\underline{b}}^{b} (v-x) \frac{dF_h(x)^2}{F_h(b)^2}$$

Simplifying

$$\Pi_l(b) = (1-\rho)^2 \left[\int_{\underline{b}}^{b} (v-x) 2F_l(x) f_l(x) dx + (1-F_l(b)^2) \left(\frac{v-w_l}{2} \right) \right] + 2\rho(1-\rho) \int_{\underline{b}}^{b} (v-x) F_h(x) f_l(x) dx + \rho^2 \int_{\underline{b}}^{b} (v-x) 2F_h(x) f_h(x) dx$$

Taking derivatives,

$$\begin{aligned} \frac{d\Pi_l(b)}{db} &= (1-\rho)^2 \left[2f_l(b)F_l(b)(v-b) - F_l(b)f_l(b)(v-w_l) \right] \\ &\quad + 2\rho(1-\rho)(v-b)F_h(b)f_l(b) + 2\rho^2(v-b)F_h(b)f_h(b) \\ &= (1-\rho)^2 f_l(b)F_l(b)(v-2b+w_l) + 2\rho(v-b)F_h(b)((1-\rho)f_l(b) + \rho f_h(b)) > 0 \end{aligned}$$

Lemma 3.6: In this case, this lemma is obvious since a low budget bidder can only win the second auction against other low budget bidders. Therefore the price the second auction will be w_l. Therefore, if $F_l(b) > 0$ for some $b < w_l$, then bidding b will be better than bidding below. Therefore, $F_l(b) = 0$.

Lemma 3.7: Again the proof is identical to the earlier proof.

Lemma 3.8: The proof for this lemma is almost identical to the original proof. The only modification we need to make is in equations (19) and (20). Under the case $w_h > 2w_l$, the payoff to a high budget bidder from competing against two low budget bidders will be $(1-\rho)^2 2(v - w_l)$, since the high budget bidder will always defeat their bids by bidding w_h or w_{h-}. The rest of the payoff functions remain the same and the proof goes through.

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