Matching with Continuous Bidirectional Investment

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Abstract

We develop a one-to-one matching game where men and women (interns and employers, etc.) exert costly efforts to produce benefits for their partners. We prove the existence and Pareto optimality of interior stable allocations, and we characterize the relationship between players' costs, efforts, benefits, and payoffs in such allocations. We find, for instance, that men and women with lower marginal costs of effort choose to provide their partners with higher benefits by exerting more effort; in return, they receive higher benefits from their partners and attain higher payoffs.

Keywords: matching with contracts, continuous contracts, stability, existence, bidirectional investment/effort, benefits produced and received, costs, sorting, and comparative statics.

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1 Introduction

Partners usually exert costly efforts to produce benefits for each other. For instance, when an (unpaid) intern and an employer match, the intern exerts effort to complete tasks that benefit the employer, while the employer exerts effort to train the intern in industry methods and practices. Likewise, following the classic example of Gale and Shapley [16], when man and a woman date, the man exerts effort to benefit his girlfriend (e.g., by cooking her dinners), while the woman exerts effort to benefit her boyfriend. Other examples include mentors and mentees, masters and apprentices (e.g., professors and graduate students), senior managers and organizations (e.g., deans and colleges), buyers and sellers of specialized goods and services (e.g., artists and patrons or lawyers and clients), and so on.

These benefits (and the associated efforts) are not exogenous. Rather, the benefit a person chooses to provide to their partner depends directly on (i) their own cost of effort and (ii) the benefit their partner provides to them. It also depends indirectly on (iii) their own and their partner's outside opportunities with the other men and women, which in turn depend on the benefits and partners chosen by these other players. For instance, if a man has a low cost of effort or receives a high benefit from his partner, then he is better positioned to exert effort on her behalf. Also, if other men desire his partner, then he needs to provide a higher benefit in order to retain her; yet, he would never choose to provide her a benefit so high as to make himself worse off than he could be with some other attainable woman. Analogous logic applies for his partner.

Our goal is to characterize how these forces shape the benefits that people produce and receive when they simultaneously choose their partners and efforts. In particular, we ask two questions. First, how does the benefit a person produces compare to the benefit they receive, i.e., the benefit their partner produces? Second, how does a person's cost of effort influence the benefits they produce and receive?

To answer these questions, we first introduce a general one-to-one matching game, called the General Game, and we prove several results concerning the existence and Pareto optimality of solutions. Subsequently, we develop a novel application of this game, called the Effort Game, in which heterogeneous men and women (interns and employers, etc.) pair with each other and exert costly efforts to benefit their partners, and we answer our main questions. Throughout, we couch the games, results, and discussions in terms of men and women with the understanding that these are only labels for the two sides.

In the General Game, a finite number of men and women pair with each other. When a man and a woman match, they select an agreement that specifies their individual and joint actions from a countable or uncountable set of feasible agreements.[1] The man's and the woman's payoffs are determined by their identities and the agreement they select. Our solution concept is a stable matching and vector of agreements, which we call a "stable allocation." In a stable allocation, (i) each player earns at least the value of being single and (ii) no man and woman can do strictly better by pairing and selecting a new agreement, i.e., no two players "block" the allocation.

We first show that a stable allocation exists when the payoffs are continuous in the agreement and the set of feasible agreements is compact (Proposition 1). The proof illuminates a connection between our game and the class of "Deferred Acceptance" algorithms. We also show that a strongly Pareto optimal stable allocation exists (Proposition 2). Subsequently, we give intuitive sufficient conditions for the existence of and the Pareto optimality of stable allocations with interior agreements (Proposition 3 and 4, respectively).

We next develop the Effort Game. In the Effort Game, when a man and a woman match, they come to an agreement about the effort each exerts. Their efforts produce benefits for each other and are chosen from a compact interval. The man's payoff is the benefit produced by the woman's effort less the cost of his own effort. Likewise, the woman's payoff is the benefit produced by the man's effort less the cost of her own effort. For simplicity, all players have the same benefit production function, which is increasing in effort.[2] Each player is also endowed with a type (e.g., ability) that affects his or her cost of effort.[3]

To address our questions, we focus on stable allocations with interior agreements. Under the natural assumptions that players dislike exerting extremely high effort and dislike exerting effort when their partners don't exert any effort, these interior stable allocations exist, are Pareto optimal, and are the only stable allocations where players are matched (Corollary 1).

As to our first question on how partners' benefits compare, we find that players match based on the benefits they produce in *any* stable allocation with interior agreement.[4] Specifically, a man who produces the l-th highest benefit among men matches to a woman who produces the l-th highest benefit among women (Proposition 5); the analogous result holds

[1] The notion that partners select agreements is natural. Since partners implicitly and explicitly choose the "rules" that govern their individual and joint actions, they effectively select an agreement. Our simplification is that these rules, or at least the important ones, are chosen at the onset of the partnership instead of over its duration.

[2] Our core results, Propositions 5 to 8, hold for heterogeneous benefit functions; see Section 4 for details. Thus, they obtain when men have one benefit function and women have a different benefit function, and so extend to settings where one side pays the other for a service, e.g., lawyers and clients.

[3] Since we place no monotonicity restrictions on the cost functions, players may benefit directly from their own efforts; see Section 4 for details.

[4] There are usually many interior stable allocations; unless mentioned otherwise, our findings pertain to all of them.

for women. Thus, (i) men who produce strictly higher benefits are matched to women who produce strictly higher benefits and (ii) men who produce the same benefit are matched to women who produce the same benefit.

The intuition for this result is that players "compete" with each other for higher benefit partners. To illustrate, suppose two men m and m' both produce the same benefit, but m' has partner w' who produces a strictly higher benefit than the partner of m. Then m finds it best to increase his effort by an arbitrarily small amount in order to produce a slightly larger benefit and "win" w' away from m'. Man m is able to win w' in this fashion because she desires the largest benefit possible. He's willing to win w' because he gains a strictly higher benefit in exchange for a smaller increase in his cost. Thus, m and w' block. It follows that a necessary condition of stability is that men who produce the same benefit are matched to women who produce the same benefit. As the intuition indicates, Proposition 5 depends on the continuity of effort and doesn't obtain with effort is discrete; see Section 5 for details.

Since the benefit production function is common across players, it follows that players match based on the effort they exert (Corollary 2). Interestingly, Rammstedt and Schupp [35] and Watson et al. [44] give empirical support for this prediction: they both find that conscientious people more frequently date other conscientious people, while lazy people more frequently date other lazy people. Our result provides a novel rationalization for this observation.

To answer our second question on how the benefits players produce and receive are shaped by their costs, we suppose that the *marginal cost* of effort is decreasing in type/ability. We find that this is sufficient to ensure that higher ability players produce larger benefits. Specifically, we show that if man m' has a strictly higher type than another man m, then m' produces a benefit at least as large as the benefit produced by m (Proposition 6); the analogous result holds for women. The intuition is that higher ability players can "outcompete" lower ability players because their lower marginal costs allow them to profitably offer slightly higher benefits. Thus, the competition for partners drives them to provide higher benefits. As the intuition suggests, Proposition 6 depends on the continuity of effort and doesn't obtain when effort is discrete; see Section 5 for details.

It follows that higher ability players (i) exert more effort (Corollary 3) and, surprisingly, (ii) receive *higher* benefits from their partners (Corollary 4). In the context of men and women, these corollaries give us an idea of how a particular kind of person may behave and fair in the dating "market." And, in the context of interns and employers, they help us understand how some employers routinely attract outstanding interns (and eventually employees). Stepping back from the model, Proposition 6 and these corollaries also provide guidance on how an employer can, via pre-match investments that reduce its marginal-cost of training, improve the rank-order caliber of intern it obtains (see the discussion after Corollary 4 in Section 5).

Under the additional assumption that all players share a common cost of zero effort, we find that weakly higher ability players have weakly higher payoffs (Proposition 7). This additional assumption ensures that higher ability players have lower costs. Thus, they profitably "imitate" and outcompete weakly lower ability players whenever such players do strictly better. Hence, a necessary condition of stability is that higher ability players earn at least as much as lower ability players. As with our previous results, Proposition 7 depends on the continuity of effort; see Section 5 for details.

Our findings suggest that higher ability players match with other higher ability players. While this need not happen in every interior stable allocation (we discuss why in Section 5), we establish that there is at least one interior stable allocation where it happens (Proposition 8). Intriguingly, Belot and Francesconi [5] and Hitsch et al. [20] find that more educated people date each other.[5] Since more educated people often have a lower cost of effort (e.g., Regan et al. [36]), their findings lend support to this prediction.

We close by examining how decreases in players' opportunity costs of effort, perhaps due to reductions in work/family responsibilities or new technologies, affect the *sizes* of players' decisions and outcomes. We proceed by supposing men and women have a symmetric endowment of types, and we select the unique "symmetric" stable allocation. (This allocation is focal because it treats equals equally and it maximizes social welfare (Lemma 10).) We find that decreases in players' marginal costs of effort, i.e., increases in their types, increase their efforts, the benefits they produce and receive, and their payoffs in this allocation (Proposition 9). The proposition follows from the symmetry of the type endowment, the submodularity of the cost function, and the fact that all players have a common cost of zero effort. Returning to our intern-employer example, this result shows that a firm can increase the *level* of benefit it obtains from its intern (and its payoff) by reducing its marginal cost of training.

The balance of this section discusses the related literature. Subsequently, Section 2 describes the General Game and stability, Section 3 gives Propositions 1 to 4, Section 4 describes the Effort Game and gives Corollary 1, and Section 5 presents Propositions 5 to 9 and Corollaries 2 to 4. Omitted proofs are relegated to the Appendix.

Related Literature

Our work makes economic and technical contributions to three literatures. The first literature examines the existence of stable allocations in matching games with agreements. We contribute to it via the General Game by weakening many of the traditional assumptions and by giving a new and general existence proof; we also contribute a novel application in

[5] Arcidiacono et al. [3] review the recent empirical literature on mate preferences.

the Effort Game. The second literature examines pre-match investment. We contribute to it via the Effort Game by allowing investment/effort to be co-determined with the matching and by characterizing the previously unexplored relationships among the benefits players produce and receive, their costs of effort, and their payoffs. The third literature examines when players match assortatively in their endowed types. We contribute to it via the Effort Game by providing simple conditions on payoffs that guarantee assortative matching in types and by focusing on the relationship between benefits and types, instead of the relationship between matched players' types.

Some background is helpful. There are three seminal games in the matching literature: the "Marriage Game" of Gale and Shapley [16]; the "Assignment Game" as articulated by Demange and Gale [12]; and the "Generalized Marriage Game" as articulated by Hatfield and Milgrom [19].[6] In the Marriage Game, men and women match with each other and receive payoffs based only on the identities of their partners. The Assignment Game generalizes the Marriage Game by allowing men and women to agree to a (real-valued) monetary transfer when they match. Thus, players' payoffs depend on their partners' identities and their transfers; these payoffs are either quasi-linear or weakly monotone in money. Both the Marriage Game and the Assignment Game are special cases of the General Game; see Example 3 (in Section 2). The Generalized Marriage Game extends the Marriage Game by allowing for many-to-one matching – e.g., a firm hiring multiple workers – and by allowing a worker and a firm to select an agreement from a *finite* set of possible agreements when they match. Agreements are more than simple transfers; in addition to salaries, they may specify vacation time, job responsibilities (e.g., teaching loads), and a myriad of factors. Hence, players' payoffs depend on the identities of their matches and their agreement. When firms can only hire one worker, the Generalized Marriage Game is a special case of the General Game.

While one might think that the Effort Game is an Assignment Game, this isn't the case. An Assignment Game only allows each matched couple to agree to a single point in $\mathbb{R}$, whereas the Effort Game allows each matched couple to agree to a point in $\mathbb{R}^2$. Thus, the Effort Game is suitable for examining the benefits that both players produce, while an Assignment Game is not. The Effort Game also makes no assumptions about the monotonicity of payoffs. Furthermore, the Effort Game is not a Generalized Marriage Game since its set of possible agreements is uncountable.

Our existence results, Propositions 1 and 2, are related to the existence results of (i) Adachi [1], Echenique and Oviedo [13], Fleiner [15], Hatfield and Milgrom [19], and Roth [37], (ii) Crawford and Knoer [11] and Quinzii [34], and (iii) Alkan and Gale [2] and Kaneko

[6] See Roth and Sotomayor [38] and Sonmez and Unver [41] for the histories of these games.

[24]. Adachi [1], Echenique and Oviedo [13], Fleiner [15], Hatfield and Milgrom [19], and Roth [37], prove existence for the Generalized Marriage Game by giving various "Deferred Acceptance" algorithms that halt at stable allocations in finite time.[7] Crawford and Knoer [11] prove existence for a quasi-linear Assignment Game by employing a contradiction argument that illuminates a connection between the class of Deferred Acceptance algorithms and the existence of stable allocations in their game. Quinzii [34] generalizes Crawford and Knoer's result by proving existence for a weakly monotone Assignment Game via Scarf's Balancedness Theorem.

Kaneko [24] considers a more abstract, one-to-one matching game between men and women. In his game, each man m and each woman w have a set of achievable payoffs V_{mw} from which they pick some point when they're matched. He shows that if this set satisfies certain conditions (e.g., if $\boldsymbol{v}' \in V_{mw}$, then $\boldsymbol{v} \leq \boldsymbol{v}'$ implies $\boldsymbol{v} \in V_{mw}$), then Scarf's Balancedness Theorem implies a stable allocation exists. Alkan and Gale [2] build on Kaneko's game by assuming each pair's set of achievable payoffs are given by their Pareto frontier. When this frontier is described by a bounded, strictly decreasing, and continuous function that intersects both the horizontal and vertical axes, they describe an algorithm that computes a stable allocation in finite time.

We differ from these studies in our assumptions and in our method of proof. Consequently, our result applies to natural games like Example 2 (in Section 2) where payoffs are non-monotone, Kaneko's [24] technical properties don't hold, and the Pareto frontier doesn't intersect either axis. Also, our method of proof is novel and it illuminates a connection between the class of Deferred Acceptance algorithms and the existence of stable allocations in the General Game. Furthermore, while many of these studies characterize general properties of the stable set (e.g., its connectedness and its lattice structure), none of them consider our main application, the Effort Game, or develop our characterizations thereof.

Speaking of the Effort Game, our findings on how players' benefits (and efforts) compare, Proposition 5 and Corollaries 1 and 2, are closely related to the rich literature on pre-match investment, including (i) Cole et al. [9, 10], Hatfield et al. [18], Mailath et al. [29], Noldeke and Samuelson [30], Peters [31], and Peters and Siow [32], (ii) Chiappori et al. [7] and Iyigun and Walsh [22], and (iii) Burdett and Coles [6] among others.[8] These studies, with

[7] These algorithms require that firms regard workers as "substitutes." While this requirement is naturally satisfied in the General Game (since it's a one-to-one game), it needn't hold in arbitrary many-to-one games and, as a rule, it's restrictive. See, for instance, Kelso and Crawford [26] and Hatfield and Milgrom [19] for discussions of the requirement, and see, for instance, Kominers and Sonmez [27], Hatfield and Kojima [17], and Sonmez and Switzer [40] for ways to weaken it.

[8] In a related work, Hoppe et al. [21] examine pre-match signaling in matching games with uncertainty and characterize when the social benefits of coordination that's facilitated by the signals outweighs their costs. These signals differ from the investments in the papers above as they are purely wasteful, i.e., generate no

the exception of Burdett and Coles [6] (which we discuss below), consider two-stage games where players first invest in themselves (e.g., go to school and the gym) and then enter a matching game. In Chiappori et al. [7], Cole et al. [9, 10], Hatfield et al. [18], and Iyigun and Walsh [22], this game is an Assignment Game where each pair receives a joint surplus that they divide with their transfer. Each pair's surplus depends on their investments and endowed characteristics, and each player's payoff is their portion of the surplus less their investment cost. In Peters [31] and Peters and Siow [32], this game is a Marriage Game where each player's payoff depends on their and their partner's investments. Noldeke and Samuelson [30] consider a more general game that's akin to Alkan and Gale's [2] game and encompasses both the Assignment and Marriage Games.

These studies examine the (Pareto) optimality/efficiency of players' investments. Cole et al. [9, 10], Mailath et al. [29], and Peters [31] find that equilibrium investments can be inefficient due to coordination failures, with both a finite and infinite number of players. In contrast, Chiappori et al. [7], Hatfield et al. [18], Iyigun and Walsh [22], and Peters and Siow [32] find that equilibrium investments are (approximately) efficient. Noldeke and Samuelson [30] explain these divergent findings by identifying general conditions under which all equilibria are efficient.

Efficiency is not the focus of Chiappori et al. [7] and Iyigun and Walsh [22]. Rather, these papers examine how labor market returns and sex ratios shape players' pre-match educational investments and the divisions of their marital surpluses. Chiappori et al. [7] show that educated men may match with uneducated women in equilibrium when men's returns to education are sufficiently high. Iyigun and Walsh [22] show that, when men are in short supply to women, then they invest less and obtain a greater proportion of the joint surplus in equilibrium than women. In addition, both studies, as well as Cole et al. [9, 10], establish that players who invest more match with each other when each pair's joint surplus is supermodular in their investments.

Burdett and Coles [6] take a different approach and consider a frictional second stage matching game, i.e., a game where players randomly encounter each other over time and, at each encounter, may either match and exit or keep searching. Each player's payoff is the quality of their partner, which depends on their partner's investment and endowment, discounted by the times spent searching less their own investment. The authors show that (i) players with higher qualities usually match with each other in equilibrium and (ii) that equilibrium investments may be inefficiently large.

Our work is complementary to these papers. Like them, we study the effects of investment. However, in the Effort Game, investment/effort is match-specific since it's determined

direct benefits for anyone.

at the time players match, whereas pre-match investments are the same for every possible match. This distinction is both intuitive and economically meaningful since it ensures, for instance, that players' investments are Pareto optimal in every solution, see Corollary 1. More importantly, we explore the relationships among the benefits that players produce and receive, their costs of effort, and their payoffs; none of the papers discussed above characterize these relationships.

To the best of our knowledge, there are no other papers that study simultaneous matching and investment in non-market environments like ours. There are, however, papers that study post match investment with commitment and information problems.[9] For instance, Kaya and Vereshchagina [25] consider a three-stage "roommates" game with transferable utility where players pair off, then each pair writes a contract that divides the profit from their joint venture, and subsequently each player decides how much costly and unobservable effort to contribute to the venture. They show that efforts/investments are (Pareto) inefficient, due to the moral hazard problem, and they develop conditions under which players match assortatively in their endowed types/productivities. Our work is complementary because we allow players to verifiably choose and compete on the basis of effort/benefit and, unlike Kaya and Vereshchagina [25], we examine the relationship between benefit and type.

Speaking of assortative matching in endowed types, our finding in this area, Proposition 8, is also related to Becker [4], Chiappori and Reny [8], and Legros and Newman [28] among others.[10] In his seminal paper, Becker [4] uses a quasi-linear Assignment Game to show that players match assortatively when each pair's joint surplus is supermodular in their types.

Chiappori and Reny [8] examine how men and women, with heterogeneous risk preferences, match to "share [the] risk" of uncertain future incomes. A man and woman share this risk by agreeing to a schedule of divisions of their future incomes when they match; the man's payoff is his expected utility-of-wealth under the schedule, likewise for the woman. Their principal finding is that matching in stable allocations is "negative assortative," i.e., that more risk-averse (higher type) men match to less risk-averse (lower type) women and vice versa.

Legros and Newman [28] consider a general, non-transferable utility matching game that's akin to the game of Alkan and Gale [2] and nests both Becker's and Chiappori and Reny's games. They develop general conditions on the Pareto frontiers of each man-woman pair

[9] If there are no commitment and information problems then simultaneous investment is the same as post-match investment.

[10] In a related work, Farrell and Scotchmer [14] study when agents with similar productivities form groups in a coalition formation game where coalitions (are forced to) divide their outputs equally. Pycia [33] builds on their paper and examines when players with similar risk preferences form groups given that coalitions divide their outputs according to exogenous sharing rules. In addition, there is a rich matching-with-frictions literature on assortative matching in types – see Smith [39] for an overview.

that guarantee that matching is either assortative or negative assortative. Our work is complementary to these studies since (i) we develop natural conditions on the primitive payoff functions under which higher types match with each other in a game with endogenous investments (and thus enriches the economic understanding of when assortative matching occurs) and (ii) we focus on the relationship between the benefit and type, instead of the relationship between matched players' endowed types.

2 Description of the General Game

This section describes the General Game, defines a stable allocation, and discusses several examples.

Environment

There are two finite sets of players, men $\mathcal{M} = \{1, \ldots, M\}$ and women $\mathcal{W} = \{M + 1, \ldots, N\}$, with $N > M > 0$. Let $\mathcal{N} = \mathcal{M} \cup \mathcal{W}$. We write m for the m-th man, w for the w-th woman, and i for the i-th player (regardless of gender).

Each player may either be single or may match with a member of the opposite sex. We adopt the convention that a single player is matched to himself or herself. A **matching** is a function that specifies each player's match, i.e., is a $\phi : \mathcal{N} \to \mathcal{N}$ such that: (i) for each man m, $\phi(m) \in \mathcal{W} \cup \{m\}$; (ii) for each woman w, $\phi(w) \in \mathcal{M} \cup \{w\}$; and (iii) for each man m and each woman w, $\phi(m) = w \iff \phi(w) = m$. We say player i is **partnered** if $\phi(i) \neq i$. We write Φ for the finite set of all matchings.

When a man and a woman match, they select an **agreement** $\boldsymbol{x} = (x_1, \ldots, x_k) \in \mathbb{R}^k$, where $k \geq 1$. Their agreement specifies their individual and joint **actions** x_1, …, x_k. For instance, x_1 and x_2 may give the number of hours m and w spend at work each week respectively, x_3 may give the number of days m and w spend camping every year, and so on. Also, each single player has an agreement $\boldsymbol{x} \in \mathbb{R}^k$ with himself or herself.

Given a $\phi \in \Phi$, we write $\boldsymbol{x}^i$ for the agreement player i has with either (i) his or her partner or (ii) himself or herself. Thus, $\boldsymbol{x}^i = \boldsymbol{x}^{\phi(i)}$ for each player i.[11] We write $\bar{\boldsymbol{x}} = (\boldsymbol{x}^1, \ldots, \boldsymbol{x}^M, \boldsymbol{x}^{M+1}, \ldots, \boldsymbol{x}^N)$ for the vector of players' agreements. Notice that $\bar{\boldsymbol{x}} \in A(\phi) = \{(\tilde{\boldsymbol{x}}^1, \ldots, \tilde{\boldsymbol{x}}^N) \in \mathbb{R}^{kN} \mid \tilde{\boldsymbol{x}}^i = \tilde{\boldsymbol{x}}^{\phi(i)} \text{ for all } i \in \mathcal{N}\}$, where $A(\phi)$ is the set of possible agreement vectors for the matching ϕ, and that $A(\phi)$ is a nonempty vector subspace of $\mathbb{R}^{kN}$. An **allocation** is a $(\phi, \bar{\boldsymbol{x}})$ such that $\phi \in \Phi$ and $\bar{\boldsymbol{x}} \in A(\phi)$, i.e., is a matching and a vector of agreements. We write $\mathcal{A} = \{(\phi, \bar{\boldsymbol{x}}) \in \Phi \times \mathbb{R}^{kN} | \bar{\boldsymbol{x}} \in A(\phi)\}$ for the set of allocations.

[11]If player i is partnered, then i and his or her match $\phi(i)$ have *an* agreement $\boldsymbol{x}$, so $\boldsymbol{x}^i = \boldsymbol{x}$ and $\boldsymbol{x}^{\phi(i)} = \boldsymbol{x}$. If player i is single, then he or she as an agreement $\boldsymbol{x}$ and $\phi(i) = i$, so $\boldsymbol{x}^{\phi(i)} = \boldsymbol{x}^i = \boldsymbol{x}$.

A player's payoff depends (only) on the identity of his or her match and their agreement. Formally, each man m has a payoff function $u_m : \{\mathcal{W} \cup \{m\}\} \times \mathbb{R}^k \to \mathbb{R}$ over his possible matches and agreements. Likewise, each woman w has a payoff function $u_w : \{\mathcal{M} \cup \{w\}\} \times \mathbb{R}^k \to \mathbb{R}$. We normalize the value of being single to zero, i.e., for every $\boldsymbol{x} \in \mathbb{R}^k$, we have $u_i(i, \boldsymbol{x}) = 0$ for each player i. Let $(\phi, \bar{\boldsymbol{x}}) = (\phi, \boldsymbol{x}^1, \ldots, \boldsymbol{x}^i, \ldots, \boldsymbol{x}^N) \in \Phi \times \mathbb{R}^{kN}$, in a slight abuse of notation we write $u_i(\phi, \bar{\boldsymbol{x}})$ for the payoff of player i in $(\phi, \bar{\boldsymbol{x}})$, i.e., $u_i(\phi, \bar{\boldsymbol{x}}) \equiv u_i(\phi(i), \boldsymbol{x}^i)$.

Stable Allocations

The next four definitions develop the idea of a stable allocation. Let $X \subset \mathbb{R}^k$ be the (nonempty) set of "feasible" agreements.[12]

Definition. An allocation $(\phi, \bar{\boldsymbol{x}})$ is *feasible* if agreements are in X, i.e., $\bar{\boldsymbol{x}} \in X^N$.

Definition. An allocation $(\phi, \bar{\boldsymbol{x}})$ is *individually rational* if every player gets at least the value of being single, i.e., $u_i(\phi, \bar{\boldsymbol{x}}) \geq 0$ for each player i.

Definition. A man m and a woman w *block* a $(\phi, \bar{\boldsymbol{x}}) \in \Phi \times \mathbb{R}^{kN}$ if they both obtain strictly higher payoffs by matching with each other at a feasible agreement than they obtain in $(\phi, \bar{\boldsymbol{x}})$, i.e., if there exists an $\boldsymbol{x} \in X$ such that

$$u_m(w, \boldsymbol{x}) > u_m(\phi, \bar{\boldsymbol{x}}) \text{ and } u_w(m, \boldsymbol{x}) > u_w(\phi, \bar{\boldsymbol{x}}).$$

Definition. An allocation $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ is *stable* if (i) it is feasible, (ii) individually rational, and (iii) no man and woman block it.

Stable allocations are our solution concept. When an allocation is stable: (i) no player can do strictly better by choosing to be single (per individual rationality) and (ii) no two players can do strictly better by matching with each other and choosing a new agreement instead of following $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ (per no blocking). As in Gale and Shapley [16], we might imagine that a stable allocation is the outcome of a bargaining process where players try to maximize their own payoffs. The rational is that no player can do strictly better by (i) opting out or (ii) by trying to strike a new bargain with some other player j, as j would reject this bargain since it doesn't make him or her strictly better off. After bargaining concludes, we imagine that players match with their agreed upon partners and take their agreed upon actions, and then receive their payoffs.[13]

[12] We take X to be the same for all men and women for notational simplicity. Our results for the General Game readily extend to the case where different man-woman pairs have different sets of feasible agreements.

[13] Generally, players may commit to their agreements because (i) there is community enforcement and players punish each other for defections (e.g., Kandori [23]), (ii) the agreements constitute enforceable contracts (e.g., a written, verbal, or "implied-in-fact" contract), (iii) they have a preference for doing so (e.g., a sense of responsibility/honor), or (iv) they maintain/gain social esteem for doing so. In the case of interns

Observe that the set of stable allocations and the core coincide because the payoffs of a matched man and woman only depend on their identities and their agreement. In addition, there are usually many stable allocations. In light of this, we focus on results about the set of stable allocations or specific selections thereof.

Examples

Since the General Game is quite abstract, it's helpful to give a few examples.

Example 1. A Simple Effort Game.

Suppose there are four players, two men and two women, i.e., $\mathcal{M} = \{1, 2\}$ and $\mathcal{W} = \{3, 4\}$. Let each player i have a type $\theta_i \in R$, e.g., innate ability. We set $\theta_1 = \theta_4 = 2$ and $\theta_2 = \theta_3 = 1$. Let $X = [0, 2]^2$ be the set of feasible agreements. When a man m and a woman w are matched (to each other), their payoffs to agreement $(x_1, x_2) \in \mathbb{R}^2$ are $u_m(w, x_1, x_2) = x_2 - \frac{(x_1)^2}{\theta_m} - 1/8$ and $u_w(m, x_1, x_2) = x_1 - \frac{(x_2)^2}{\theta_w} - 1/8$. Recall that single players get zero.

For $(x_1, x_2) \in X$, we think of x_1 and x_2 as the man's and the woman's efforts respectively. Thus, both players exert costly effort to make each other happy: m exerts effort x_1 to produce a benefit of x_1 for w and incurs a cost of $(x_1)^2/\theta_m + \frac{1}{8}$ for doing so, while w exerts effort x_2 to produce a benefit of x_2 for m and incurs a cost of $(x_2)^2/\theta_w + \frac{1}{8}$ for doing so. (The assumption that each player's effort is in $[0, 2]$ reflects the idea that people can only work so hard as they face time and energy limitations.)

One stable allocation of this game is $(\phi^\star, \bar{\boldsymbol{x}}^\star) = (\phi^\star, \boldsymbol{x}^{1^\star}, \boldsymbol{x}^{2^\star}, \boldsymbol{x}^{3^\star}, \boldsymbol{x}^{4^\star})$, where $\phi^\star(1) = 4$ and $\boldsymbol{x}^{1^\star} = \boldsymbol{x}^{4^\star} = (1, 1)$, and $\phi^\star(2) = 3$ and $\boldsymbol{x}^{2^\star} = \boldsymbol{x}^{3^\star} = (1/2, 1/2)$. That is, man 1 matches with woman 4 and they both exert effort 1, while man 2 matches with woman 3 and they both exert effort $1/2$.

Let's verify $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ is stable. To do this, we need to show that it's feasible, individually rational, and not blocked. Feasibility is automatic since $(1, 1)$ and $(1/2, 1/2)$ are in X. Individual rationality requires a bit more work. We have $u_1(\phi^\star, \bar{\boldsymbol{x}}^\star) = 1 - 1/2(1)^2 - 1/8 = 3/8$. By analogous calculations $u_2(\phi^\star, \bar{\boldsymbol{x}}^\star) = 1/8$, $u_3(\phi^\star, \bar{\boldsymbol{x}}^\star) = 1/8$, and $u_4(\phi^\star, \bar{\boldsymbol{x}}^\star) = 3/8$. It follows that $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ is individually rational, i.e., no player can do better by choosing to be single.

We need to make sure that there are no blocking pairs. The definition of blocking makes no assumptions about which players block an allocation: players who are matched may block, as may players who are not matched. Thus, we need to check four pairs.

Consider man 1 and woman 3. They block if there is a $(x_1, x_2) \in X$ such that $u_1(3, x_1, x_2) > 3/8$ and $u_3(1, x_1, x_2) > 1/8$, i.e., if $x_2 - \frac{1}{2}(x_1)^2 > \frac{1}{2}$ and $x_1 - (x_2)^2 > \frac{1}{4}$. Since it's readily verified

and firms, commitment is natural because of (ii); in fact, firms have been sued for not adequately training their interns (e.g., the 2013 case of Glatt et al. vs. Fox Searchlight Pictures). In the case of men and women, commitment is less natural, but still justifiable as an approximation: early in their relationships couples often behave in ways that indicate their long-term conduct, so we may think of them as selecting goals/an agreement and trying to follow through.

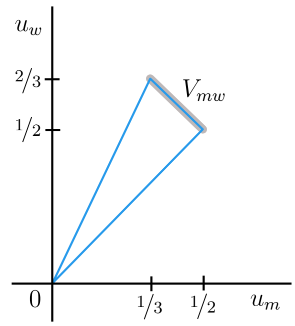

Figure 2.1: Achievable Payoffs in Example 2 for Man m and Woman w

that this system has no real solution, they can't block, i.e., they can't do better by matching and choosing a new agreement instead of following $(\phi^\star, \bar{\boldsymbol{x}}^\star)$. Symmetry gives that man 2 and woman 4 can't block either.

Consider man 1 and woman 4. They block if there is a $(x_1, x_2) \in X$ such that $u_1(4, x_1, x_2) > 3/8$ and $u_4(1, x_1, x_2) > 3/8$, i.e., if $x_2 - \frac{1}{2}(x_1)^2 > \frac{1}{2}$ and $x_1 - \frac{1}{2}(x_2)^2 > \frac{1}{2}$. Since this system has no real solution, they can't block. An analogous argument gives that man 2 and woman 3 can't block. It follows that $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ is stable. △

Example 2. Hiking.

Let $\mathcal{M} = \{1, 2\}$, let $\mathcal{W} = \{3, 4\}$, and let $X = [0, 1]$. When a man m and a woman w are matched, their payoffs to agreement $x \in \mathbb{R}$ are

$$u_m(w, x) = \begin{cases} x & \text{if } x \leq \frac{1}{2} \\ 1 - x & \text{if } x > \frac{1}{2} \end{cases} \text{ and } u_w(m, x) = \begin{cases} 2x & \text{if } x \leq \frac{1}{3} \\ 1 - x & \text{if } x > \frac{1}{3}, \end{cases}$$

Recall that single players get zero. For $x \in X$, we think of x as the percentage of their Sundays m and w spend hiking – w only wants to go for a third of the day, while m wants to go for half the day.

Let V_{mw} be the set of payoffs that man m and woman w can achieve when matched, i.e., $V_{mw} = \{(v_m, v_w) \in \mathbb{R}^2 | u_m(w, x) = v_m \text{ and } u_w(m, x) = v_w \text{ for some } x \in X\}$. This set is the blue triangle plotted in Figure 2.1; it's bounded and has an empty interior. The Pareto frontier is the northeastern edge of V_{mw}; it's highlighted in gray in the figure. While the frontier is strictly decreasing, it never intersects the horizontal or vertical axes. Thus, V_{mw} does not satisfy the assumptions of Alkan and Gale [2] or Kaneko [24].

Nonetheless, there are a continuum of stable allocations. One set is described by $(\phi^\star, \bar{\boldsymbol{x}}^\star)$, where $\phi^\star(1) = 3$, $\phi^\star(2) = 4$, and $\bar{\boldsymbol{x}}^\star = (x, x, x, x)$ with $x \in [\frac{1}{3}, \frac{1}{2}]$. That is, man 1 matches with woman 3, man 2 matches with woman 4, and all players hike for a common proportion of their Sunday x, with $x \in [\frac{1}{3}, \frac{1}{2}]$.

Such an allocation is trivially feasible and individually rational. In such an allocation, (a) man 1 and woman 3 or (b) man 1 and woman 4 can't block as an increase in the duration makes woman 3 or woman 4 worse off and a decrease in the duration makes man 1 worse off. Likewise, (c) man 2 and woman 3 or (d) man 2 and woman 4 can't block. △

As previously mentioned, the General Game nests both the Marriage and Assignment Games. Since payoffs in the Marriage Game only depend on the identities of players' matches, we can embed it in the General Game by taking X to be a singleton. The next example illustrates the embedding of a simple, quasi-linear Assignment Game.

Example 3. A Quasi-Linear Assignment Game.

Recall that in a quasi-linear Assignment Game, men and women receive some surplus from matching, which they split via a monetary transfer. For each man m and each woman w, let $f_m(w) \geq 0$ give m's surplus from matching with w and let $f_w(m) \geq 0$ give w's surplus from matching with m. When m and w match, they agree to a transfer $x \in \mathbb{R}$; m earns his surplus less the transfer x and w earns her surplus plus x. Single players earn nothing.

To embed this game in the General Game, first let each man m and each woman w's payoffs to matching with agreement $x \in \mathbb{R}$ be $u_m(w, x) = f_m(w) - x$ and $u_w(m, x) = f_w(m) + x$. (Recall that single player automatically get zero.) Next, let the set of feasible agreements be $X = [-\bar{\alpha}, \bar{\alpha}]$, where $\bar{\alpha} = \max_{(m,w)\in\mathcal{M}\times\mathcal{W}}\{f_m(w) + f_w(m)\}$. (This is without loss, if a man and woman have a transfer that's not in X, then one of them has a negative payout. Thus, in any individually rational – and, by inclusion, any stable allocation – all transfers are in X.[14]) This completes the embedding since it's easily verified that a stable allocation satisfies Demange and Gale's [12] definition of stability. △

Remark. While the General Game subsumes both the Marriage and Assignment Games, it doesn't preserve many of their properties – e.g., there is usually no "man-preferred" stable allocation because of indifference over agreements. We discuss this in the Supplement.

3 Results for the General Game

In this section, we state and prove our main results for the General Game; we develop several additional results in the Supplement. The proofs are deferred to the end of each subsection in order to discuss the results.

Existence of Stable Allocations

[14]Broadly, Demange and Gale's [12] they require that, for each man m and each woman w, there is a finite α'_{mw} such that if m and w's transfer is not in $[-\alpha'_{mw}, \alpha'_{mw}]$, then either m or w has a negative payoff. Thus, letting $X = [-\bar{\alpha}', \bar{\alpha}']$, where $\bar{\alpha}' = \max_{m,w}\{\alpha'_{mw}\}$, is without loss.

In this subsection, we prove that stable allocations exist under the following assumption.

Assumption 1. Compactness and Continuity.
The set of feasible agreements X is compact and, for each man m and each woman w, the payoffs $u_m(w', \boldsymbol{x})$ and $u_w(m', \boldsymbol{x})$ are continuous in $\boldsymbol{x}$ for all women w' and for all men m' respectively.

Proposition 1. Existence of a Stable Allocation.
Let Assumption 1 hold, then a stable allocation $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ exists.

Proposition 1 gives, for instance, that there is a stable allocation in Examples 1 and 2, in the (Generalized) Marriage Game, and in the Assignment Game since inspection shows that Assumption 1 holds in each. We prove Proposition 1 by contradiction. Specifically, we show that if there is no stable allocation when X is compact, then there is no stable allocation when agreements are restricted to a finite subset of X. This contradicts a well-known result (see Lemma 1 below) that there is always a stable allocation when the set of feasible agreements is finite.[15] The key insight of our proof is that under the contradiction hypothesis and Assumption 1, we can use the Heine-Borel Theorem to ensure the existence of this finite subset of X. We need four preliminary results to make this argument precise. (Also, we remark on ways to weaken Assumption 1 after the proof.)

Lemma 1. Finite Existence.
Let the set of feasible agreements X be finite, then a stable allocation $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ exists.

Results like Lemma 1 are standard in the one-to-one and many-to-one matching literatures when payoffs are such that players can strictly order matches and agreements; for instance, Hatfield and Milgrom's Theorem 3 [19] and Roth's Theorem 1 [37]. Both papers prove existence by giving variants of Gale and Shapley's [16] Deferred Acceptance algorithm that find a stable allocation in finite time. The problem is that players in our game may be indifferent. Fortunately, these algorithms can be modified to allow for indifferences via the inclusion of a "tie-breaking" rule – see Roth and Sotomayor's Theorem 2.8 [38] for an example of this kind of modification – so the lemma obtains. (In the Supplement, we give a proof of Lemma 1 that's based on tie-breaking.)

Lemma 2. Continuity of $u_i(\phi, \bar{\boldsymbol{x}})$.
Let Assumption 1 hold and let $\phi \in \Phi$, then $u_i(\phi, \bar{\boldsymbol{x}})$ is continuous in $\bar{\boldsymbol{x}}$ for each player i.

Proof. This is a direct consequence of Assumption 1. □

[15] The intuition here is similar to that of Crawford and Knoer [11]; however, our game, assumptions, and formal approach are quite different from theirs.

To prove Proposition 1, we need to (i) represent the set of feasible and individually rational allocations as a collection of compact sets and (ii) establish that the set of allocations a man and woman block with a given agreement is open. To these ends, we introduce some notation. Let $\phi \in \Phi$ and let $F(\phi) = \{\bar{\boldsymbol{x}} \in \mathbb{R}^{kN} | \, \bar{\boldsymbol{x}} \in X^N \cap A(\phi) \text{ and } u_i(\phi, \bar{\boldsymbol{x}}) \geq 0 \text{ for all } i \in \mathcal{N}\}$ be the (possibly empty) set of agreement vectors in $\mathbb{R}^{kN}$ such that the pair $(\phi, \bar{\boldsymbol{x}})$ is a feasible and individually rational allocation for each $\bar{\boldsymbol{x}} \in F(\phi)$. Let $\Phi_F = \{\phi \in \Phi | F(\phi) \neq \emptyset\}$ be the set of matchings such that, for each $\phi \in \Phi_F$, there is an $\bar{\boldsymbol{x}}$ so that $(\phi, \bar{\boldsymbol{x}})$ is feasible and individually rational. Clearly, an allocation $(\phi, \bar{\boldsymbol{x}})$ is feasible and individually rational if an only if $\phi \in \Phi_F$ and $\bar{\boldsymbol{x}} \in F(\phi)$.

Lemma 3. Compactness of $F(\phi)$.
Let Assumption 1 hold and let $\phi \in \Phi$, then $F(\phi)$ is compact.

Proof. See the Appendix. □

When a man m and a woman w block a $(\phi, \bar{\boldsymbol{x}}) \in \Phi \times \mathbb{R}^{kN}$ with agreement $\boldsymbol{x} \in X$, we say $(m, w, \boldsymbol{x})$ **blocks** $(\phi, \bar{\boldsymbol{x}})$. Let $C = \mathcal{M} \times \mathcal{W} \times X$. For a $\phi \in \Phi$ and $c = (m, w, \boldsymbol{x}) \in C$, let $D_\phi(c) = \{\bar{\boldsymbol{x}} \in \mathbb{R}^{kN} | \, u_m(w, \boldsymbol{x}) > u_m(\phi, \bar{\boldsymbol{x}}) \text{ and } u_w(m, \boldsymbol{x}) > u_w(\phi, \bar{\boldsymbol{x}})\}$ be the set of vectors in $\mathbb{R}^{kN}$ such that c blocks the pair $(\phi, \bar{\boldsymbol{x}})$ for each $\bar{\boldsymbol{x}} \in D_\phi(c)$.

Lemma 4. Openness of $D_\phi(c)$.
Let Assumption 1 hold, let $\phi \in \Phi$, and let $c \in C$, then $D_\phi(c)$ is open.

Proof. See the Appendix. □

Proof of Proposition 1.[16] Suppose that there is no stable allocation. Since the set of feasible and individually rational allocations is non-empty (because players may always be matched to themselves), every feasible and individually rational allocation is blocked by some man and woman. Let $\phi \in \Phi_F$. Then, for every $\bar{\boldsymbol{x}} \in F(\phi)$, we have that $(\phi, \bar{\boldsymbol{x}})$ is blocked by a $c \in C$, which implies that $F(\phi) \subset \cup_{c \in C} D_\phi(c)$. Since $D_\phi(c)$ is open by Lemma 4, we have $\{D_\phi(c)\}_{c \in C}$ is an open cover of $F(\phi)$. Since $F(\phi)$ is compact by Lemma 3, the Heine-Borel Theorem gives the existence of a finite sub-cover $\{D_\phi(c_{\phi j})\}_{j=1}^{l_\phi}$. Thus, for every $\bar{\boldsymbol{x}} \in F(\phi)$, we have $(\phi, \bar{\boldsymbol{x}})$ is blocked by some element of $\{c_{\phi j}\}_{j=1}^{l_\phi}$.

Repeating this argument for all matchings in Φ_F gives a set $E = \cup_{\phi \in \Phi_F} \{c_{\phi j}\}_{j=1}^{l_\phi}$ such that every feasible and individually rational allocation is blocked by an element of E. Since Φ is finite, Φ_F and thus E are finite. Let $E_X = \{\boldsymbol{x} \in X \, | \, (m, w, \boldsymbol{x}) \in E \text{ for some } (m, w) \in \mathcal{M} \times \mathcal{W}\}$ be the set of agreements associated with E. Observe that E_X is finite.

To establish the contradiction, suppose that the set of feasible agreements is E_X instead of X. Since E_X is finite, Lemma 1 gives that there is a stable allocation $(\phi^\star, \bar{\boldsymbol{x}}^\star)$. By the

[16] This proof benefited greatly from discussions with Asaf Plan.

definition of stability, (i) $(\phi^\star, \bar{\boldsymbol{x}}^\star) \in \mathcal{A}$, (ii) $\bar{\boldsymbol{x}}^\star \in (E_X)^N$, (iii) $u_i(\phi^\star, \bar{\boldsymbol{x}}^\star) \geq 0$ for each player i, and (iv) there is no $(m, w, \boldsymbol{x}) \in \mathcal{M} \times \mathcal{W} \times E_X$ such that $u_m(w, \boldsymbol{x}) > u_m(\phi^\star, \bar{\boldsymbol{x}}^\star)$ and $u_w(w, \boldsymbol{x}) > u_w(\phi^\star, \bar{\boldsymbol{x}}^\star)$. Since $(E_X)^N \subset X^N$, we have $\bar{\boldsymbol{x}}^\star \in X^N$, which, in light of (i) and (iii), implies that $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ is a feasible and individually rational allocation when the set of feasible agreements is X. Thus, the previous paragraph gives that there is a $(m', w', \boldsymbol{x}') \in E$ such that $u_{m'}(w', \boldsymbol{x}') > u_{m'}(\phi^\star, \bar{\boldsymbol{x}}^\star)$ and $u_{w'}(w', \boldsymbol{x}') > u_{w'}(\phi^\star, \bar{\boldsymbol{x}}^\star)$. Since $\boldsymbol{x}' \in E_X$, we have that $(m', w', \boldsymbol{x}') \in \mathcal{M} \times \mathcal{W} \times E_X$. Hence, (iv) is contradicted, i.e., the display equation gives that m' and w' block $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ when the set of feasible agreements is E_X, a contradiction of stability. □

Remark. It's possible to weaken a few of our assumptions and still obtain existence. In particular, we may (i) allow different men and women to have different sets of feasible agreements, (ii) allow for heterogeneous values to being single (or remove the option to be single), and (iii) replace continuity with upper-semicontinuity (or even weaker continuity assumptions). We may also generalize the agreement space to be a closed subset of a compact metric space. We cannot, however, easily dispense with the compactness of X. We discuss these points in the Supplement.

Pareto Optimality

In this subsection, we first make the notion of a Pareto optimal allocation precise and then we show that Assumption 1 also ensures the existence of Pareto optimal stable allocations. Such allocations are important for social welfare and have suggested as an alternative solution concept in two-sided matching games (e.g., Sotomayor [42]).

Definition. An allocation $(\phi, \bar{\boldsymbol{x}})$ is *Pareto optimal* if there is no other feasible allocation $(\phi', \bar{\boldsymbol{x}}')$ such that (i) all players do weakly better in $(\phi', \bar{\boldsymbol{x}}')$ than $(\phi, \bar{\boldsymbol{x}})$, i.e., $u_i(\phi', \bar{\boldsymbol{x}}') \geq u_i(\phi, \bar{\boldsymbol{x}})$ for each player i, and (ii) at least one player does strictly better in $(\phi', \bar{\boldsymbol{x}}')$ than $(\phi, \bar{\boldsymbol{x}})$, i.e., $u_i(\phi', \bar{\boldsymbol{x}}') > u_i(\phi, \bar{\boldsymbol{x}})$ for some player i. If a stable allocation $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ is Pareto optimal, we say it is a Pareto stable allocation.

We focus on "strong" Pareto optimality instead of "weak" Pareto optimality since every stable allocation is weakly Pareto optimal and since strong Pareto optimality is more intuitive. (Notice that Pareto optimal allocations won't generally maximize social welfare because payoffs are not quasi-linear.)

Proposition 2. Existence of a Pareto Stable Allocation.
Let Assumption 1 hold, then a Pareto stable allocation $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ exists.

Proposition 2 gives, for instance, that there is a Pareto stable allocation in Examples 1 and 2, in the Marriage Game, and in the Assignment Game. We prove Proposition 2 by showing that the set of stable allocations is "compact" given Assumption 1 (see Lemma 5 below). Thus, there is an allocation $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ that maximizes total welfare on this set (see Lemma 6 below). We then argue that $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ is Pareto optimal via contradiction. Indeed, if it weren't, then there would be another feasible allocation $(\phi', \bar{\boldsymbol{x}}')$ where all players do weakly better and one player does strictly better. But then, $(\phi', \bar{\boldsymbol{x}}')$ is stable (see Lemma 7 below). Hence, $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ does not maximize total welfare on the set of stable allocations, a contradiction. We develop our three preliminary lemmas before making this argument precise.

Let $\mathcal{S}$ denote the set of stable allocations. We need to represent $\mathcal{S}$ as collection of compact subsets. To these ends, for each $\phi \in \Phi$, let $\mathcal{S}(\phi) = \{\bar{\boldsymbol{x}} \in \mathbb{R}^{kN} | (\phi, \bar{\boldsymbol{x}}) \in \mathcal{S}\}$ be the (possibly empty) set of agreement vectors that yield a stable allocation when paired with ϕ. Let $\Phi_{\mathcal{S}} = \{\phi \in \Phi | \, \mathcal{S}(\phi) \neq \emptyset\}$ be set of matchings which are part of a stable allocation. An allocation $(\phi, \bar{\boldsymbol{x}})$ is stable if and only if $\phi \in \Phi_{\mathcal{S}}$ and $\bar{\boldsymbol{x}} \in \mathcal{S}(\phi)$.

Lemma 5. Compactness of $\mathcal{S}(\phi)$.
Let Assumption 1 hold, then $\mathcal{S}(\phi)$ is compact for each $\phi \in \Phi$.

Proof. See the Appendix. □

Given a $(\phi, \bar{\boldsymbol{x}}) \in \Phi \times \mathbb{R}^{kN}$, let $T(\phi, \bar{\boldsymbol{x}}) = \Sigma_{i \in \mathcal{N}} u_i(\phi, \bar{\boldsymbol{x}})$ be the total payoff to all players in $(\phi, \bar{\boldsymbol{x}})$.

Lemma 6. Total Welfare and Stable Allocations.
Let Assumption 1 hold, then there is a stable allocation $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ such that $T(\phi^\star, \bar{\boldsymbol{x}}^\star) \geq T(\phi', \bar{\boldsymbol{x}}')$ for every $(\phi', \bar{\boldsymbol{x}}') \in \mathcal{S}$.

Proof. See the Appendix. □

Let $(\phi, \bar{\boldsymbol{x}})$ be a feasible allocation and let $P(\phi, \bar{\boldsymbol{x}}) = \{(\phi', \bar{\boldsymbol{x}}') \in \mathcal{A} | \, \bar{\boldsymbol{x}}' \in X^N \text{ and } u_i(\phi', \bar{\boldsymbol{x}}') \geq u_i(\phi, \bar{\boldsymbol{x}}) \text{ for all } i \in \mathcal{N}\}$ be the set of feasible allocations that are at least as good as $(\phi, \bar{\boldsymbol{x}})$. This set is non-empty as $(\phi, \bar{\boldsymbol{x}}) \in P(\phi, \bar{\boldsymbol{x}})$.

Lemma 7. A Representation of Pareto Optimality.
L7.1 : Let $(\phi', \bar{\boldsymbol{x}}')$ be a stable allocation, then every element of $P(\phi', \bar{\boldsymbol{x}}')$ is also a stable allocation.
L7.2 : Let $(\phi', \bar{\boldsymbol{x}}')$ be a stable allocation and let $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ be a solution to $\max_{(\phi, \bar{\boldsymbol{x}}) \in P(\phi', \bar{\boldsymbol{x}}')} T(\phi, \bar{\boldsymbol{x}})$. Then $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ is a Pareto stable allocation.

Proof. See Appendix A. □

Proof of Proposition 2. By Lemma 6, there is a $(\phi^\star, \bar{\boldsymbol{x}}^\star) \in \mathcal{S}$ with $T(\phi^\star, \bar{\boldsymbol{x}}^\star) \geq T(\phi, \bar{\boldsymbol{x}})$ for all $(\phi, \bar{\boldsymbol{x}}) \in \mathcal{S}$. Since $(\phi^\star, \bar{\boldsymbol{x}}^\star) \in P(\phi^\star, \bar{\boldsymbol{x}}^\star)$ and since $P(\phi^\star, \bar{\boldsymbol{x}}^\star) \subset \mathcal{S}$ per Lemma 7, we have that $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ solves $\max_{(\phi,\bar{\boldsymbol{x}})\in P(\phi^\star,\bar{\boldsymbol{x}}^\star)} T(\phi, \bar{\boldsymbol{x}})$. Thus, Lemma 7 gives that $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ is a Pareto stable allocation. □

Interior Stable Allocations

In this subsection, we first make the notion of an interior allocation precise and then we give conditions that ensure the existence of interior stable allocations. These allocations are important from a technical perspective because our proofs in Section 5 leverage small increases and decreases in effort, which are impossible at boundary allocations.

For a set $S \subset \mathbb{R}^k$, we write ∂S for the boundary of S and $\text{int}(S)$ for the interior of S.

Definition. An allocation $(\phi, \bar{\boldsymbol{x}}) = (\phi, \boldsymbol{x}^1, \dots, \boldsymbol{x}^N)$ is *interior* if some player is partnered, i.e., $\phi(i) \neq i$ for some $i \in \mathcal{N}$, and all partnered players have agreements on the interior of X, i.e., $\phi(i) \neq i$ implies $\boldsymbol{x}^i \in \text{int}(X)$ for all $i \in \mathcal{N}$. If a (Pareto) stable allocation is interior, we say it is an interior (Pareto) stable allocation.

The next assumption is sufficient for the existence of interior stable allocations.

Assumption 2. Sufficient Conditions for an Interior Stable Allocation.
The set of feasible agreements X has nonempty interior and:

1. There is a man m, a woman w, and an agreement $\boldsymbol{x} \in \text{int}(X)$ such that $u_m(w, \boldsymbol{x}) \geq 0$ and $u_w(m, \boldsymbol{x}) \geq 0$.
2. For each man m, each woman w, and each $\boldsymbol{x} \in \partial X$, least one of the following holds:
   (a) Either $u_m(w, \boldsymbol{x}) < 0$ or $u_w(m, \boldsymbol{x}) < 0$.
   (b) There is a $\boldsymbol{x}' \in \text{int}(X)$ such that $u_m(w, \boldsymbol{x}') > u_m(w, \boldsymbol{x})$ and $u_w(m, \boldsymbol{x}') > u_w(m, \boldsymbol{x})$.

Part (1) ensures that being partnered is not worse than being single for at least one man and one woman; we refer to this as **agreeability**. Part (2) gives that each boundary agreement is either (a) very undesirable to the man or the woman and/or (b) payoff dominated by some other feasible agreement. If (a) holds for all men, all women, and all $\boldsymbol{x} \in \partial X$, then we say **intolerability** holds. If (b) holds for all men, all women, and all $\boldsymbol{x} \in \partial X$, then we say that **blockability** holds.
Part (i) ensures that being partnered is not worse than being single for at least one man and one woman; we refer to this as **agreeability**. Part (ii) gives that each boundary agreement is either (a) very undesirable to the man or the woman and/or (b) payoff dominated by some other feasible agreement. If (a) holds for all men, all women, and all $\boldsymbol{x} \in \partial X$, then we say **intolerability** holds. If (b) holds for all men, all women, and all $\boldsymbol{x} \in \partial X$, then we say that **blockability** holds.

While intolerability and blockability are strong assumptions, they are often reasonable. In fact, we make heavy use of intolerability in the next section because it embeds the intuitive ideas that a player is very unhappy when (i) he or she exerts extreme effort β or (ii) he or she exerts positive effort and his or her match does not; see the continuation of Example 1 below and see Section 4.

Proposition 3. Existence of an Interior Stable Allocation.
Let Assumptions 1 and 2 hold. Then, (i) an interior Pareto stable allocation $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ *exists and (ii) any stable allocation where some player is partnered is an interior stable allocation.*

Proposition 3 gives, for instance, that there is an interior Pareto stable allocation in Examples 1 and 2 since Assumptions 1 and 2 hold. It's easily seen that Assumption 1 hold in both examples, so we only need to verify that Assumption 2 holds. We do this next by showing that agreeability and intolerability/blockability hold.

Example 1 (Continued). Intolerability.

We want to verify that agreeability and intolerability hold. We have that agreeability holds since $u_1(4, \boldsymbol{x}) = 3/8 > 0$ and $u_4(1, \boldsymbol{x}) = 3/8 > 0$ when $\boldsymbol{x} = (1, 1)$.

Showing intolerability takes a bit more work: for each man m and each woman w, we need to walk the boundary of $[0, 2]^2$ and show that either m or w has a negative payoff. Focus on m and recall that $u_m(w, x_1, x_2) = x_2 - \frac{1}{\theta_m}(x_1)^2 - 1/8$. For all $y \in [0, 2]$, we have that (i) $u_m(w, y, 0) = -1/8 - \frac{1}{\theta_m} y^2 < 0$, i.e., m gets a negative payout when he exerts effort and w does not, and (ii) $u_m(w, 2, y) = y - 1/8 - \frac{4}{\theta_m} < 0$ since $\theta_w \in \{1, 2\}$, i.e., m gets a negative payout if he exerts extreme effort. By symmetry, we have (iii) $u_w(m, 0, y) < 0$ for and (iv) $u_w(m, y, 2) < 0$ for all $y \in [0, 2]$. Thus, for every $(x_1, x_2) \in \partial[0, 2]^2$, we have either $u_m(w, x_1, x_2) < 0$ or $u_w(m, x_1, x_2) < 0$. $\triangle$

Example 2 (Continued). Blockability.

We want to verify that this example satisfies agreeability and blockability. We have that agreeability holds since $u_m(w, x) = 0$ and $u_w(m, x) = 0$ for each man m and each woman w when $x = 0$. We also have that blockability holds as m and w's payoffs are strictly increasing at $x = 0$ and strictly decreasing at $x = 1$. $\triangle$

We prove Proposition 3 in two steps. First, we use a contradiction argument to establish that any stable allocation with a partnered player is an interior stable allocation. The key insight is that if a matched man and woman have a boundary agreement then Assumption 2 ensures either (i) the agreement isn't individually rational or (ii) they can block. In either case, we obtain the requisite contradiction. In the second step, we leverage this result and a construction argument to show the existence of an interior Pareto stable allocation.

Proof of Proposition 3. We begin by establishing that a stable allocation $(\phi^\star, \bar{\boldsymbol{x}}^\star) = (\phi^\star, \boldsymbol{x}^{1^\star}, \ldots, \boldsymbol{x}^{N^\star})$ where some player is partnered is interior. That is, we show that that $\phi^\star(i) \neq i$ implies $\boldsymbol{x}^{i^\star} \in \text{int}(X)$ for each player i. Without loss, consider a man m with

$\phi^\star(m) \neq m$, the argument is analogous for a woman. Let $w = \phi^\star(m)$. We argue by contradiction. If $\boldsymbol{x}^{m^\star} \in \partial X$, then, since Assumption 2 holds, we have: (a) $u_m(w, \boldsymbol{x}^{m^\star}) < 0$ or $u_w(m, \boldsymbol{x}^{m^\star}) < 0$ or (b) there is a $\boldsymbol{x}' \in X$ such that $u_m(w, \boldsymbol{x}') > u_m(w, \boldsymbol{x}^{m^\star})$ and $u_w(m, \boldsymbol{x}') > u_w(m, \boldsymbol{x}^{m^\star})$. If (a), we have that $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ is not individually rational, a contradiction. If (b), m and w block $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ with agreement $\boldsymbol{x}'$, another contradiction. It follows that $\boldsymbol{x}^{m^\star} \in \text{int}(X)$.

We now show the existence of an interior Pareto stable allocation. By Proposition 2, there is a Pareto stable allocation $(\phi^\star, \bar{\boldsymbol{x}}^\star) = (\phi^\star, \boldsymbol{x}^{1^\star}, \ldots, \boldsymbol{x}^{N^\star})$. There are two cases (i) some man and woman are matched by $\phi^\star$ or (ii) no man and woman are matched by $\phi^\star$. If case (i), then $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ is an interior allocation by the above paragraph; so we proceed under (ii).

By Assumption 2, there is a man m, a woman w, and an agreement $\tilde{\boldsymbol{x}} \in \text{int}(X)$ such that m and w both get at least zero at agreement $\tilde{\boldsymbol{x}}$. We construct a candidate interior Pareto stable allocation $(\phi', \bar{\boldsymbol{x}}') = (\phi', \boldsymbol{x}^{1'}, \ldots, \boldsymbol{x}^{N'})$ from $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ by taking m and w, matching them, and assigning them agreement $\tilde{\boldsymbol{x}}$, while leaving everyone else single – i.e., by setting $\phi'(m) = w$, $\phi'(w) = m$, $\boldsymbol{x}^{m'} = \tilde{\boldsymbol{x}}$, set $\boldsymbol{x}^{w'} = \tilde{\boldsymbol{x}}$, and $\phi'(i) = \phi^\star(i)$ and $\boldsymbol{x}^{i'} = \boldsymbol{x}^{i^\star}$ for all $i \in \mathcal{N} \backslash \{m, w\}$. Clearly, $(\phi', \bar{\boldsymbol{x}}')$ is an interior allocation. Since each player i gets zero in $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ and gets at least zero in $(\phi', \bar{\boldsymbol{x}}')$, Lemma 7 gives that $(\phi', \bar{\boldsymbol{x}}')$ is stable. It remains to prove that $(\phi', \bar{\boldsymbol{x}}')$ is Pareto optimal. To these ends, suppose it were not. Then there is another feasible allocation $(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger)$ such that (i) $u_i(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger) \geq u_i(\phi', \bar{\boldsymbol{x}}')$ for all $i \in \mathcal{N}$ and (ii) $u_i(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger) > u_i(\phi', \bar{\boldsymbol{x}}')$ for some $i \in \mathcal{N}$. Since $u_i(\phi', \bar{\boldsymbol{x}}') \geq u_i(\phi^\star, \bar{\boldsymbol{x}}^\star)$ for all $i \in \mathcal{N}$, (i) and (ii) imply that $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ is not Pareto optimal, a contradiction. □

Interior Stable Allocations and Pareto Optimality

In this subsection, we give conditions that guarantee *every* interior stable allocation is Pareto optimal. These conditions tie the last two subsections together and allow us to ensure that every solution of the Effort Game is Pareto optimal.

We need a definition. We write $B_r(\boldsymbol{x})$ for the open ball of radius r in $\mathbb{R}^k$ around $\boldsymbol{x}$, i.e., $B_r(\boldsymbol{x}) = \{\boldsymbol{x}' \in \mathbb{R}^k \mid ||\boldsymbol{x} - \boldsymbol{x}'|| < r\}$.

Definition. For a man m and a woman w, we say that $u_m(w, \boldsymbol{x})$ (or $u_w(m, \boldsymbol{x})$) is *locally nonsatiated in* $\boldsymbol{x}$ if, for each agreement $\boldsymbol{x} \in \mathbb{R}^k$, there is a nearby agreement that m (w) strictly prefers, i.e., there is a $\boldsymbol{x}' \in B_r(\boldsymbol{x})$ such that $u_m(w, \boldsymbol{x}) < u_m(w, \boldsymbol{x}')$ for each $r > 0$ (a $\boldsymbol{x}'' \in B_r(\boldsymbol{x})$ such that $u_w(m, \boldsymbol{x}) < u_w(m, \boldsymbol{x}'')$ for each $r > 0$).

The next assumption gives our sufficient conditions, the proposition follows.

Assumption 3. Conditions for Interior Stable Allocations to be Pareto Optimal.
The set of feasible agreements X has nonempty interior, intolerability holds, and, for each man m and each woman w, the payoffs $u_m(w', \boldsymbol{x})$ and $u_w(m', \boldsymbol{x})$ are continuous and locally

nonsatiated in $\boldsymbol{x}$ for all women w' and for all men m' respectively.

Proposition 4. Interior Stable Allocations are Pareto Optimal.
Let Assumption 3 hold, then every interior stable allocation $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ *is Pareto optimal.*

Proposition 4 gives, for instance, that the stable allocation we found in Example 1 is Pareto optimal. Simply, for each man m and each woman w, we have that $u_m(w, x_1, x_2)$ and $u_w(m, x_1, x_2)$ are strictly increasing in either x_1 or x_2 and so are locally nonsatiated. We prove Proposition 4 by using a "bribery" argument. The idea is that local nonsatiation allows a man (woman) to bribe an indifferent woman (man) to form a blocking pair. Thus, if a stable allocation is not Pareto optimal, the player who does strictly better in the improving allocation bribes his or her match in the improving allocation into blocking the initial stable allocation, a contradiction.

Proof of Proposition 4. We argue by contradiction. Suppose that $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ is an interior stable allocation that's not Pareto optimal. Then there is another feasible allocation $(\phi', \bar{\boldsymbol{x}}') = (\phi', \boldsymbol{x}^{1'}, \ldots, \boldsymbol{x}^{N'})$ with (i) $u_i(\phi', \bar{\boldsymbol{x}}') \geq u_i(\phi^\star, \bar{\boldsymbol{x}}^\star)$ for all $i \in \mathcal{N}$ and (ii) $u_i(\phi', \bar{\boldsymbol{x}}') > u_i(\phi^\star, \bar{\boldsymbol{x}}^\star)$ for some $i \in \mathcal{N}$. Suppose that man m is the player who does strictly better; this is without loss as the analogous argument applies when a woman does strictly better. We'll show that m and some woman block $(\phi^\star, \bar{\boldsymbol{x}}^\star)$. But first, we need two preliminary facts.

Fact One: m is matched to some woman under ϕ'. If not, then $u_m(\phi', \bar{\boldsymbol{x}}') = 0$. Since $u_m(\phi', \bar{\boldsymbol{x}}') > u_m(\phi^\star, \bar{\boldsymbol{x}}^\star)$, we have $u_m(\phi^\star, \bar{\boldsymbol{x}}^\star) < 0$, a contradiction of the stability of $(\phi^\star, \bar{\boldsymbol{x}}^\star)$. In light of this, let $w = \phi'(m)$.

Fact Two: m's agreement $\boldsymbol{x}^{m'}$ is on the interior of X. If not, then $\boldsymbol{x}^{m'} \in \partial X$, so either $u_m(\phi', \bar{\boldsymbol{x}}') = u_m(w, \boldsymbol{x}^{m'}) < 0$ or $u_w(\phi', \bar{\boldsymbol{x}}') = u_w(m, \boldsymbol{x}^{m'}) < 0$ by Assumption 3. Since $u_i(\phi', \bar{\boldsymbol{x}}') \geq u_i(\phi^\star, \bar{\boldsymbol{x}}^\star)$ for all $i \in \mathcal{N}$, either $u_m(\phi^\star, \bar{\boldsymbol{x}}^\star) < 0$ or $u_w(\phi^\star, \bar{\boldsymbol{x}}^\star) < 0$, a contradiction as $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ is stable.

We now establish that m and w block $(\phi^\star, \bar{\boldsymbol{x}}^\star)$, i.e., we establish that there is an $\boldsymbol{x} \in X$ such that (i) $u_m(w, \boldsymbol{x}) > u_m(\phi^\star, \bar{\boldsymbol{x}}^\star)$ and (ii) $u_w(m, \boldsymbol{x}) > u_w(\phi^\star, \bar{\boldsymbol{x}}^\star)$. Since payoffs are continuous in $\boldsymbol{x}$ by Assumption 3 and since $u_m(w, \boldsymbol{x}^{m'}) > u_m(\phi^\star, \bar{\boldsymbol{x}}^\star)$, there is an $r > 0$ such that equation (i) holds for all $\boldsymbol{x} \in B_r(\boldsymbol{x}^{m'})$. Since $\boldsymbol{x}^{m'} \in \text{int}(X)$, we may take r to be sufficiently small such that $B_r(\boldsymbol{x}^{m'}) \subset X$. Recall that $u_w(m, \boldsymbol{x}^{m'}) = u_w(\phi', \bar{\boldsymbol{x}}') \geq u_w(\phi^\star, \bar{\boldsymbol{x}}^\star)$. Since $u_w(\cdot)$ is locally nonsatiated, there is an $\boldsymbol{x} \in B_r(\boldsymbol{x}^{m'})$ with $u_w(m, \boldsymbol{x}) > u_w(m, \boldsymbol{x}^{m'})$. Thus, equations (i) and (ii) hold at $\boldsymbol{x}$, i.e., m and w block $(\phi^\star, \bar{\boldsymbol{x}}^\star)$, a contradiction. □

Remark. We cannot meaningfully swap blockability for intolerability in Assumption 3. Simply, blockability and local nonsatiation are mutually exclusive conditions when X is compact and we usually take X to be compact.[17]

4 Description of the Effort Game

In this section, we describe the Effort Game and show that it has a solution.

Environment

Let $\Theta \subset \mathbb{R}_+$ be a finite set of types ordered in the usual way. We endow each player i with a type θ_i (e.g., innate ability) from Θ, and suppose that types are commonly known. Let $b : \mathbb{R} \to \mathbb{R}$ be a continuous and strictly increasing function. We call $b(\cdot)$ the "benefit" function because if a player exerts effort y, then he or she provides benefit $b(y)$ to his or her match. Let $c : \Theta \times \mathbb{R} \to \mathbb{R}$ be continuous in its second argument. We call $c(\cdot)$ the "cost" function because if a type θ player exerts effort y, he or she incurs cost $c(\theta, y)$. Unlike the benefit function, the cost function may be non-monotone in effort. For simplicity, we take $b(0) \geq 0$ and $c(\theta, 0) \geq 0$ for all types θ. We suppose that, due to time and energy limitations, players can only exert efforts between 0 and β, where $0 < \beta < \infty$. Thus, the set of feasible agreements/efforts is $X = [0, \beta]^2$.

When a man m and a woman w are matched (to each other), their payoffs from agreement $(x_1, x_2) \in \mathbb{R}^2$ are

$$u_m(w, x_1, x_2) = b(x_2) - c(\theta_m, x_1) \text{ and } u_w(m, x_1, x_2) = b(x_1) - c(\theta_w, x_2).$$

Recall that single players get zero. As in Example 1, for $(x_1, x_2) \in X$, we think of x_1 and x_2 as the man's and the woman's efforts respectively. Thus, m and w exert effort to produce a benefit for each other.

It may be the case that a player's effort diminishes his or her payoff, i.e., $c(\theta, y)$ is positive and increasing in y. However, a player's effort often provides him or her with a "personal" benefit, e.g., in cooking a delicious dinner for his girlfriend, a man also cooks himself a delicious dinner. We capture this personal benefit by allowing $c(\theta, y)$ to be non-monotone in y and so be decreasing and even negative.

The benefit function is homogenous for simplicity. Proposition 5 continues to hold when (i) the benefit a player produces depends on his/her identity or (ii) payoffs are *not* additively separable, but are continuous and are strictly increasing in benefit. Propositions 6, 7, and

[17] To see this, suppose local nonsatiation holds, blockability holds, payoffs are continuous in $\boldsymbol{x}$, and X is compact. Let m be a man, and let w be a woman. The Extreme Value Theorem gives that $\alpha = \arg\max_{\boldsymbol{x} \in X} u_m(w, \boldsymbol{x})$ is nonempty. Since blockability gives that every point in ∂X is payoff dominated by a point in $\text{int}(X)$, we have $\alpha \subset \text{int}(X)$. But then, for each $\boldsymbol{x} \in \alpha$, there is a $B_r(\boldsymbol{x}) \subset X$ for some $r > 0$, so local nonsatiation gives that there is a $\boldsymbol{x}' \in B_r(\boldsymbol{x})$ with $u_m(w, \boldsymbol{x}') > u_m(w, \boldsymbol{x})$. Thus, $\boldsymbol{x} \notin \alpha$, a contradiction.

8 continue to hold when the benefit a player produces depends on his or her (i) sex/side or (ii) type, *provided* this benefit is increasing in his or her type. Thus, our core results extend to environments where one side pays the other for a service, e.g., lawyers and clients.[18] We discuss why each result generalizes after we present it in Section 5.

We make a clarifying remark before proceeding.

Remark. Since players prefer matches who produce higher benefits, it may appear as though they have a common preference (over matches and agreements). This is not the case.[19] Instead, for a fixed vector of efforts, each side agrees on a ranking of the opposite side – e.g., the women agree on which men are best, second best, and so on. However, efforts aren't fixed, they're endogenous. Thus, each man effectively chooses his position in the women's ranking by choosing his effort, likewise for women. (His choice depends, of course, on the benefits offered by the other men and women, as well as his own cost.)

Interior Stable Allocations

We focus on interior stable allocation since we often maintain the following assumption.

Assumption 4. Agreeability and Intolerability.
The benefit and cost functions, as well as players' endowed types are such that agreeability and intolerability hold.

This assumption embeds the intuitive ideas that a player is very unhappy when (i) he or she exerts extreme effort β or (ii) he or she exerts positive effort and his or her match does not. The class of benefit functions, cost functions, and endowed types satisfying Assumption 4 is nonempty – e.g., Example 1, where $b(y) = y$ and $c(\theta, y) = {}^{y^2}/_{\theta} + {}^1/_8$, as well as Examples 4 and 5 below. Other examples can be readily constructed since agreeability only requires $b(x_2) \geq c(\theta_m, x_1)$ and $b(x_1) \geq c(\theta_w, x_2)$ for some man m, woman w, and $(x_1, x_2) \in X$, while intolerability only requires $c(\theta_i, \beta) > b(\beta)$ and $c(\theta_i, y) > b(0)$ for all $y \in [0, \beta]$ and each player i.[20]

The next corollary motivates our focus on interior stable allocations.

Corollary 1. Existence and Pareto Optimality of Interior Stable Allocations.

[18] In such a game, the clients' benefit and cost functions are linear in money/effort, while the lawyers' benefit and cost functions are non-linear in effort.

[19] To illustrate, consider two men m and m' who have a choice between (a) woman w and agreement $(x_1, x_2) = ({}^1/_4, {}^9/_{10})$ or (b) woman w' and agreement $(x'_1, x'_2) = ({}^1/_2, 1)$. Also, let $b(y) = y$, let $c(\theta_m, y) = y^2 - y + {}^1/_2$, and let $c(\theta_{m'}, y) = (2y)^2 - y + {}^1/_4$. A bit of algebra shows that m strictly prefers (b) to (a), while m' strictly prefers (a) to (b).

[20] Assumption 4 prevents players from obtaining a *positive* payoff from their own efforts since the cost function cannot be negative. Nonetheless, the cost function can be decreasing (e.g., $c(\theta, y) = (1 - x^2 + \frac{1}{2}x^3)/\theta$ on $\mathbb{R}_+$) so players can benefit from their own efforts. Notice also that Assumption 4 is inessential to Propositions 5 to 7, so these results hold even when players obtain positive payoffs from their own efforts.

Let Assumption 4 hold, then (i) there is a interior stable allocation $(\phi^\star, \bar{\boldsymbol{x}}^\star)$, (ii) all interior stable allocations are Pareto optimal, and (iii) any stable allocation where a player is partnered is an interior stable allocation.

Proof. Since $b(y)$ and $c(\theta, y)$ are continuous, $b(y)$ is strictly increasing in y (and, thus, locally nonsatiated), X is compact with nonempty interior, and agreeability and intolerability hold, both Propositions 3 and 4 apply. $\square$

Our second motivation for focusing on interior stable allocations is technical: they allow us to consider the effects of small shifts in effort. Such shifts lie at the heart of our proofs in the next section and are impossible at boundary allocations.

5 Results for the Effort Game

In this section, we state and prove our results for the Effort Game. We defer the proofs of the major results until the end of each subsection in order to discuss the results and provide economic intuition for them.

Matching on the Basis of Benefit and Effort

In this subsection, we examine how players match on the basis of their benefits and efforts. To do this, we need some notation. Let $(\phi^\star, \bar{\boldsymbol{x}}^\star) = (\phi^\star, \boldsymbol{x}^{1\star}, \ldots, \boldsymbol{x}^{N\star})$ be an interior stable allocation. Let $z_i^\star$ denote player i's effort in $(\phi^\star, \bar{\boldsymbol{x}}^\star)$, i.e., $z_i^\star$ is the first component of $\boldsymbol{x}^{i\star} = (x_1^{i\star}, x_2^{i\star})$ when i is a man and $z_i^\star$ is the second component of $\boldsymbol{x}^{i\star}$ when i is a woman. Consider players who are partnered by $\phi^\star$. Each partnered player i produces a positive benefit $b_i^\star = b(z_i^\star)$. We rank partnered players by the benefits they produce from greatest to least and place them into groups of equivalent benefits.[21] For the men, we label these groups $G_1^{\mathbb{M}}$, …, $G_{J_{\mathbb{M}}}^{\mathbb{M}}$, where $G_1^{\mathbb{M}}$ contains the men who produce the highest benefit, $G_2^{\mathbb{M}}$ contains the men who produce the second highest benefit, and so on. For the women, we label the analogous groups $G_1^{\mathbb{W}}$, …, $G_{J_{\mathbb{W}}}^{\mathbb{W}}$. We have that $J_{\mathbb{M}}$ and $J_{\mathbb{W}}$ are positive and finite, since at least one man and woman are partnered and since $\mathcal{M}$ and $\mathcal{W}$ are finite. To simplify notation, we suppress the dependence of $\{z_i^\star\}_{i\in\mathcal{N}}$, $\{b_i^\star\}_{i\in\{j\in\mathcal{N}|\phi^\star(j)\neq j\}}$, $\{G_l^{\mathbb{M}}\}_{l=1}^{J_{\mathbb{M}}}$, $\{G_l^{\mathbb{W}}\}_{l=1}^{J_{\mathbb{W}}}$, $J_{\mathbb{M}}$, and $J_{\mathbb{W}}$ on $(\phi^\star, \bar{\boldsymbol{x}}^\star)$.

The next proposition characterizes how players match across benefit groups.

Proposition 5. Benefit Groups and Matching.
Let $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ be an interior stable allocation and let $G_1^{\mathbb{M}}, \ldots, G_{J_{\mathbb{M}}}^{\mathbb{M}}$ and $G_1^{\mathbb{W}}, \ldots, G_{J_{\mathbb{W}}}^{\mathbb{W}}$ be the associated benefit groups. Then, (i) there are an equal number of male and female benefit

[21] We omit single players because they produce benefits for no one.

groups, i.e., $J_{\mathbb{M}} = J_{\mathbb{W}}$*, and (ii) a man* m *is in the* l*-th benefit group of men if and only if his match* $\phi^{\star}(m)$ *is in the* l*-th benefit group of women, i.e.,* $m \in G_l^{\mathbb{M}} \iff \phi^{\star}(m) \in G_l^{\mathbb{W}}$ *for all* $l \in \{1, \ldots, J_{\mathbb{M}}\}$*. The analogous result holds for women.*

That is, in any interior stable allocation, (i) men who produce strictly higher benefits are matched to women who produce strictly higher benefits and (ii) men who produce the same benefit are matched to women who produce the same benefit. Hence, we say that matching is "assortative in benefit." We can see this in the stable allocation we found in Example 1.

Example 1 (Continued). Assortative in Benefit.

In the stable allocation we found, man 1 and woman 4 each exert effort 1, while man 2 and woman 3 each exert effort 1/2. It follows that, $b_1^{\star} = b_4^{\star} = 1$ and $b_2^{\star} = b_3^{\star} = \frac{1}{2}$. Thus, there are two (singleton) benefit groups on each side: $G_1^{\mathbb{M}} = \{1\}$, $G_2^{\mathbb{M}} = \{2\}$, $G_1^{\mathbb{W}} = \{4\}$, and $G_2^{\mathbb{W}} = \{3\}$. Since man 1 is matched to woman 4 and man 2 is matched to woman 3, the matching is assortative in benefit. $\triangle$

The intuition behind Proposition 5 is that players "compete" for the best possible match. To illustrate, suppose that there is an interior stable allocation where two partnered men m and m' both produce the same benefit, but m' has match w' who produces more benefit than the match of m. Then, m can increase his effort by an arbitrarily small amount and give w' a higher benefit than she is currently receiving. Since w' desires the highest benefit possible, she'll agree to match with m instead of m'. Man m is willing to increase his effort slightly because, in exchange for an arbitrarily small change in cost, he gains a partner with strictly higher benefit, and so does strictly better. Thus, m and w' block. Hence, a necessary condition of stability is that all men of the same benefit are matched to women of the same benefit. The proof makes this intuition precise.

Since the intuition (and thus the proof) do not depend on players having the same benefit function and do not exploit the additive nature of payoffs, the proposition continues to hold when (i) the benefits players produce depend on their identities or (ii) players' payoffs are non-separable, but are continuous and strictly increasing in the benefits they receive. Interestingly, the proposition also obtains when the benefit function is non-monotone and cost is strictly increasing since we can slightly reduce the effort of w' instead of increasing the effort of m. Unfortunately, it fails when both benefit and cost are non-monotone as then a small change in effort may not make a player strictly better off.

As the intuition implies, the continuity of effort is essential to Proposition 5. If effort is discrete, then the smallest effort increment may be so large that the benefit m gains from matching with w' is dissipated by the increase in his cost from his additional effort. Hence, the proposition may fail. For example, when $\mathcal{M} = \{1, 2\}$, $\mathcal{W} = \{3, 4\}$, $b(y) = 1 + y$,

$c(\theta, y) = {}^{y}/_{\theta}$, $\theta_1 = \theta_2 = \theta_3 = \theta_4 = 1$, and the set of feasible agreements is $\{0, 1\}^2$, then it's stable for man 1 and woman 3 to match with agreement $(0, 1)$ and for man 2 and woman 3 to match with agreement $(1, 1)$. The men clearly produce different benefits and yet are matched to women who produce the same benefit. Nevertheless, a "weak version" of the proposition holds when X is discrete: in any stable allocation $(\phi^\star, \bar{\boldsymbol{x}}^\star)$, the match of any man in $G_j^{\mathbb{M}}$ produces a weakly higher benefit than the match of any man in $G_{j+1}^{\mathbb{M}}$; likewise for women.

Since the benefit function is increasing, an implication of Proposition 5 is that players in higher benefit groups exert more effort than those in lower benefit groups. In particular, men in $G_1^{\mathbb{M}}$ exert the most effort among men, men in $G_2^{\mathbb{M}}$ exert the second most effort among men, and, in general, men in $G_l^{\mathbb{M}}$ exert the l-th most effort among men. Analogously, women in $G_l^{\mathbb{M}}$ exert the l-th most effort among women. The next corollary formalizes this result.

Corollary 2. Effort and Matching.
Let $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ be an interior stable allocation. Then, a man m exerts the l-th most effort among men if and only if his match $\phi^\star(m)$ exerts the l-th most effort among women for all $l \in \{1, \ldots, J_{\mathbb{M}}\}$. The analogous result holds for women.

Proof. Obvious and omitted. □

The corollary gives that, in any interior stable allocation, the hardest working players match, as do the second hardest working players, and so forth. This is easily seen in the interior stable allocation we found in Example 1.

We prove Proposition 5 by induction. First, we use a competition argument to establish that men in the first benefit group match with women in the first benefit group and vice-versa. Subsequently, we use induction to show that the analogous result holds for the second benefit groups, the third benefit groups, and so on.

Proof of Proposition 5. Let $\{z_i^\star\}_{i \in \mathcal{N}}$ and $\{b_i^\star\}_{i \in \{j \mid \phi^\star(j) \neq j\}}$ be the associated efforts and benefits of $(\phi^\star, \bar{\boldsymbol{x}}^\star)$.

Let $\underline{J} = \min\{J_{\mathbb{M}}, J_{\mathbb{W}}\}$ and $\overline{J} = \max\{J_{\mathbb{M}}, J_{\mathbb{W}}\}$. We show below that, for each $1 \leq j \leq \underline{J}$, we have $m \in G_l^{\mathbb{M}} \iff \phi^\star(m) \in G_l^{\mathbb{W}}$ for all $l \in \{1, \ldots, j\}$. We refer to this as the "induction" result since we prove it by induction on j: we first establish that it holds when $j = 1$; subsequently, we show that if it holds at $j - 1$, then it also holds at j when $j > 1$.

The first part of the proposition follows from the induction result. Simply, if $\underline{J} = J_{\mathbb{M}} < J_{\mathbb{W}} = \overline{J}$, then the women in group $J_{\mathbb{M}} + 1$ are matched to men *not* in groups 1 to $J_{\mathbb{M}}$, a contradiction as these groups contain all partnered men. Analogously, $\underline{J} = J_{\mathbb{W}} < J_{\mathbb{M}} = \overline{J}$ leads to a contradiction. Thus, we have $\underline{J} = J_{\mathbb{M}} = J_{\mathbb{W}} = \overline{J}$. The second part of the proposition also follows from the induction result, take $j = J_{\mathbb{M}}$.

Let $j = 1$. There are two cases: (i) $\overline{J} = 1$ and (ii) $\overline{J} > 1$. If case (i), then $\underline{J} = 1$ and the induction result is trivially true. Thus, we proceed under case (ii).

We establish that $m \in G_1^{\mathbb{M}}$ implies $\phi^\star(m) \in G_1^{\mathbb{W}}$. We do this by contradiction. Let $m \in G_1^{\mathbb{M}}$ and suppose that $w = \phi^\star(m)$ is in $G_{l'}^{\mathbb{W}}$ with $l' > 1$. Let w' be an arbitrary woman in $G_1^{\mathbb{W}}$ and let $m' = \phi^\star(w')$. We have (a) that m' may or may not be in $G_1^{\mathbb{M}}$ and (b) that $b^\star_{w'} - b^\star_w > 0$. We now establish that m and w' block $(\phi^\star, \bar{\boldsymbol{x}}^\star)$.

Man m and woman w' block with agreement $(x_1, z^\star_{w'})$ if there is an $x_1 \in [0, \beta]$ such that (i) $u_{w'}(m, x_1, z^\star_{w'}) > u_{w'}(\phi^\star, \bar{\boldsymbol{x}}^\star)$ and (ii) $u_m(w', x_1, z^\star_{w'}) > u_m(\phi^\star, \bar{\boldsymbol{x}}^\star)$. Since $z^\star_{w'}$ is the same on both sides of equation (i), the equation reduces to (i') $b(x_1) > b^\star_{m'}$. Since w' gives m benefit $b^\star_{w'}$ at $(x_1, z^\star_{w'})$, equation (ii) reduces to (ii') $b^\star_{w'} - b^\star_w > c(\theta_m, x_1) - c(\theta_m, z^\star_m)$.

There are two sub-cases. If $m' \notin G_1^{\mathbb{M}}$, then $b(z^\star_m) > b^\star_{m'}$. Thus, $x_1 = z^\star_m$ satisfies equations (i') and (ii') as $b^\star_{w'} - b^\star_w > 0$. Since $z^\star_m \in (0, \beta)$, m and w' block $(\phi^\star, \bar{\boldsymbol{x}}^\star)$, a contradiction. If $m' \in G_1^{\mathbb{M}}$, consider $x_1 = z^\star_m + \delta$, with $\delta > 0$. Since $b(z^\star_m) = b^\star_{m'}$ and $b(y)$ is strictly increasing in y, we have that $z^\star_m + \delta$ satisfies equation (i'). Since $c(\theta, y)$ is continuous in y and since $b^\star_{w'} - b^\star_w > 0$, we also have that $z^\star_m + \delta$ satisfies equation (ii') for δ sufficiently small. Since $z^\star_m \in (0, \beta)$, we can shrink δ such that $z^\star_m + \delta \in [0, \beta]$. Thus, m and w' block $(\phi^\star, \bar{\boldsymbol{x}}^\star)$, a contradiction.

The analogous argument gives that $w \in G_1^{\mathbb{M}}$ implies $\phi^\star(w) \in G_1^{\mathbb{M}}$. Since $\phi^\star(m) = w$ if and only if $\phi^\star(w) = m$, we have $m \in G_l^{\mathbb{M}} \iff \phi^\star(m) \in G_l^{\mathbb{W}}$ for all $l \in \{1\}$, i.e., the induction result is true when $j = 1$.

Let $1 < j \leq \underline{J}$ and assume that $m \in G_l^{\mathbb{M}} \iff \phi^\star(m) \in G_l^{\mathbb{W}}$ for all $l \in \{1, \ldots, j-1\}$. By the induction hypothesis, it suffices to show that $m \in G_j^{\mathbb{M}} \iff \phi^\star(m) \in G_j^{\mathbb{W}}$ to prove the induction result. Again, there are two cases: (i) $j = \overline{J}$ or (ii) $j < \overline{J}$. If case (i), then we're done. Simply, when $m \in G_j^{\mathbb{M}}$, the induction hypothesis gives he is *not* matched to a woman in a lower-indexed group. Since there is no higher index group of women, we necessarily have that $\phi^\star(m) \in G_j^{\mathbb{W}}$. Analogously, if $w \in G_j^{\mathbb{W}}$, then $\phi^\star(m) \in G_j^{\mathbb{M}}$. It follows that $m \in G_j^{\mathbb{M}} \iff \phi^\star(m) \in G_j^{\mathbb{W}}$. Thus, we proceed under case (ii).

We establish that $m \in G_j^{\mathbb{M}}$ implies $\phi^\star(m) \in G_j^{\mathbb{W}}$. As before, we do this by contradiction. Let $m \in G_j^{\mathbb{M}}$ and suppose that $w = \phi^\star(m)$ is in $G_{l'}^{w}$ with $l' \neq j$. The induction hypothesis gives that $l' > j$. Let w' be an arbitrary woman in $G_j^{\mathbb{W}}$ and let $m' = \phi^\star(w')$. Again, m' may or may not be in $G_j^{\mathbb{M}}$; but, the induction hypothesis gives that $m' \in G_{l'}^{\mathbb{M}}$ with $l' \geq j$. We (again) establish that m and w' block $(\phi^\star, \bar{\boldsymbol{x}}^\star)$.

Repeating the same argument as above gives that m and w' block with agreement $(x_1, z^\star_{w'})$ if there is an $x_1 \in [0, \beta]$ such that (i) $b(x_1) > b^\star_{m'}$ and $b^\star_{w'} - b^\star_w > c(\theta_m, x_1) - c(\theta_m, z^\star_m)$. If $m' \notin G_j^{\mathbb{M}}$, then $x_1 = z^\star_m$ satisfies equations (i) and (ii) is in $[0, \beta]$. If $m' \in G_j^{\mathbb{M}}$, consider $x_1 = z^\star_m + \delta$, with $\delta > 0$. Since m and m' are in the same benefit group and $b(y)$ is strictly increasing, we have that $z_m^\star + \delta$ satisfies equation (i). Since $c(\theta, y)$ is continuous in y and $b_{w'}^\star - b_w^\star > 0$, we have that equation (ii) holds for δ sufficiently small. Since $z_m^\star \in (0, \beta)$, we can shrink δ such that $z_m^\star + \delta \in [0, \beta]$. In either case, m and w' block $(\phi^\star, \bar{\boldsymbol{x}}^\star)$, a contradiction.

The analogous argument gives that $w \in G_j^{\mathbb{W}}$ implies $\phi^\star(w) \in G_j^{\mathbb{M}}$. It follows that $m \in G_j^{\mathbb{M}} \iff \phi^\star(m) \in G_j^{\mathbb{W}}$. □

The Relationships Among Benefits, Efforts, and Types

In this subsection, we link types to costs with the next assumption and then we characterize the relationships among the benefits players produce and receive, their efforts, and their types. We also discuss the rank-order effects of changes in types.

Assumption 5. Cost and Type.
Higher type players have strictly lower incremental costs of effort, i.e., $c(\theta, y') - c(\theta, y)$ is strictly decreasing in θ for all $y < y'$.

That is, $c(\cdot)$ is strictly submodular, as is the case when the *marginal* cost of effort is strictly decreasing in type. The next proposition examines the relationship between type and benefit produced.

Proposition 6. Benefit Produced and Type.
Let Assumption 5 hold, let $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ be an interior stable allocation, and let $G_1^{\mathbb{M}}, \ldots, G_{J_{\mathbb{M}}}^{\mathbb{M}}$ and $G_1^{\mathbb{W}}, \ldots, G_{J_{\mathbb{M}}}^{\mathbb{W}}$ be the associated benefit groups. Then, strictly higher type men produce weakly higher benefits, i.e., if two men m and m' are partnered, then $\theta_m < \theta_{m'}$ implies $m \in G_l^{\mathbb{M}}$ and $m' \in G_j^{\mathbb{M}}$ with $j \leq l$. The analogous result holds for women.[22]

Observe that (i) this result applies to every interior stable allocation and (ii) implies that players in higher benefit groups have weakly higher types.[23] We can easily see this relationship in the stable allocation we found in Example 1 since the marginal cost $2y/\theta$ is strictly decreasing in θ.

Example 1 (Continued). Benefit and Type.

Recall (i) that man 1 and woman 4 are the high types, with $\theta_1 = \theta_4 = 2$, (ii) that man 2 and woman 3 are the low types, with $\theta_2 = \theta_3 = 1$, and (iii) $b_1^\star = b_4^\star = 1$ and $b_2^\star = b_3^\star = 1/2$. Thus, the higher types produce higher benefits. The higher types also receive higher benefits

[22] We give and discuss a stronger result in the Supplement: strictly higher type men produce *strictly* higher benefits, i.e., $\theta_m < \theta_{m'}$ implies $m \in G_l^{\mathbb{M}}$ and $m' \in G_j^{\mathbb{M}}$ with $j < l$, when (i) Assumption 5 holds, (ii) $b(y)$ and $c(\theta, y)$ are continuously differentiable, and (iii) $\partial b(y)/\partial y > 0$ and $\partial c(\theta, y)/\partial y > 0$ for all $y \in [0, \beta]$.

[23] In particular, for the men, the contraposition of Proposition 6 implies that $\min\{\theta_m | m \in G_l^{\mathbb{M}}\} \geq \max\{\theta_m | m \in G_{l+1}^{\mathbb{M}}\}$ for any $l \in \{1, \ldots, J_{\mathbb{M}} - 1\}$. That is, the lowest type man in the l-th benefit group has a type at least as high as the highest type man in the $l+1$-th benefit group. The analogous result holds for women.

from their partners (since 1 and 4 are match and 2 and 3 are matched) and exert more effort (since $b(y) = y$). △

The intuition for the proposition is that higher types can "outcompete" lower types because their lower marginal costs allow them to profitably offer slightly higher benefits. Thus, the competition for matches drives them to offer higher benefits. To be more precise, suppose there is an interior stable allocation $(\phi^\star, \bar{\boldsymbol{x}}^\star)$, with benefits $\{b_i^\star\}_{i \in \{j | \phi^\star(j) \neq j\}}$ and efforts $\{z_i^\star\}_{i \in \mathcal{N}}$, where a lower type man m produces a greater benefit than a higher type man m', i.e., where $b_{m'}^\star < b_m^\star$. Let w be the match of m and let w' be the match of m'. Since the benefit function is the same for both men, w' does strictly better with m whenever he exerts more effort than m'. Thus, stability requires that m does not do better with w' at such an effort, implying $b_w^\star - c(\theta_m, z_m^\star) \geq b_{w'}^\star - c(\theta_m, z_{m'}^\star)$. Since m' has a strictly lower marginal cost than m and since $z_{m'}^\star < z_m^\star$ (because $b_{m'}^\star < b_m^\star$), we have $b_w^\star - b_{w'}^\star > c(\theta_{m'}, z_m^\star) - c(\theta_{m'}, z_{m'}^\star)$. That is, m' does strictly better by offering w agreement $(z_m^\star, z_w^\star)$ then he does in $(\phi^\star, \bar{\boldsymbol{x}}^\star)$. Hence, he's willing to outcompete m by exerting slightly more effort and offering a slightly higher benefit to win w, implying that m' and w block. It follows that a necessary condition of stability is that higher type men produce higher benefits. The proof makes this intuition precise.

Since the intuition (and thus the proof) do not depend on men *and* women having the same benefit function, the proposition readily generalizes to the case where each side has its own benefit function. In fact, the proposition continues to hold when the benefit a player produces is increasing in his or her type and effort.[24]

As the intuition suggests, the continuity of effort is essential for Proposition 6. When efforts are discrete, lower types may produce *strictly* higher benefits. For example, when $\mathcal{M} = \{1, 2\}$, $\mathcal{W} = \{3, 4\}$, $b(y) = y$, $c(\theta, y) = y/\theta$, $\theta_1 = \theta_3 = 1$ and $\theta_2 = \theta_4 = 2$, and the set of feasible agreements is $\{0, 1/2, 1\}^2$, then it's stable for man 1 and woman 4 to match with agreement $(1, 1)$ and for man 2 and woman 3 to match with agreement $(1/2, 1/2)$. Herein, man 2 produces a lower benefit despite having a higher type.

Also essential for Proposition 6 is the fact that m' has a strictly higher type than m. If m and m' have the type, then m' may not have incentive to outcompete m – formally, we cannot ensure $b_w^\star - c(\theta_m, z_m^\star) \geq b_{w'}^\star - c(\theta_m, z_{m'}^\star)$ implies $b_w^\star - b_{w'}^\star > c(\theta_{m'}, z_m^\star) - c(\theta_{m'}, z_{m'}^\star)$. Thus, players of the same type may produce different benefits, as the next example illustrates.

Example 4. Benefit and Type for a Variant of Example 1.

[24] There are two key parts to the intuition: (i) $b_{m'}^\star < b_m^\star$ implies $z_{m'}^\star < z_m^\star$ and (ii) $b(\theta_{m'}, z_m^\star) \geq b(\theta_m, z_m^\star)$, where $b(\theta, y)$ denotes the type-dependent benefit function. Part (i) ensures that m' does strictly better by matching with w and exerting slightly more effort than m, while part (ii) ensures that w does strictly better by matching with m' when he exerts slightly more effort than m. Since both (i) and (ii) are hold when $b(\theta, y)$ is increasing in both arguments, m and w' block and the proposition obtains.

Suppose $X = [0,2]^2$. Let $\mathcal{M} = \{1,2\}$ and let $\mathcal{W} = \{3,4\}$. Let $b(y) = y$ and let $c(\theta, y) = \frac{1}{\theta}y^2$. Let $\theta_1 = \theta_2 = \theta_3 = 2$ and let $\theta_4 = 1$. It's readily verified that one stable allocation is for (i) man 1 and woman 3 to match with agreement $(1,1)$ and for (ii) man 2 and woman 4 to match with agreement $(q, \frac{1}{2}(1+q^2)) \approx (0.682, 0.733)$, where $q = ((18 + 2\sqrt{93})^{1/3} - 2(\frac{\sqrt{93}-9}{4})^{1/3}) \cdot 6^{-2/3}$. Although both men are of the same type (and Assumption 5 holds), they clearly produce different benefits. △

Two corollaries of Proposition 6 follow. The first concerns the relationship between type and effort, while the second concerns the relationship between type and the benefit received.

Corollary 3. Effort and Type.
Let Assumption 5 hold, let $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ be an interior stable allocation, and let $\{z_i^\star\}_{i\in\mathcal{N}}$ be the associated efforts. Then, strictly higher type men exert weakly more effort, i.e., if two men m and m' are partnered, then $\theta_m < \theta_{m'}$ implies that $z_m^\star \leq z_{m'}^\star$. The analogous result holds for women.

Proof. A direct implication of Proposition 6 since $b(y)$ is strictly increasing. □

Corollary 4. Benefit Received and Type.
Let Assumption 5 hold, let $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ be an interior stable allocation, and let $G_1^{\mathbb{M}}, \ldots, G_{J_{\mathbb{M}}}^{\mathbb{M}}$ and $G_1^{\mathbb{W}}, \ldots, G_{J_{\mathbb{M}}}^{\mathbb{W}}$ be the associated benefit groups. Then, strictly higher type men are matched to women in weakly higher benefit groups, i.e., if two men m and m' are partnered, then $\theta_m < \theta_{m'}$ implies $\phi^\star(m) \in G_l^{\mathbb{W}}$ and $\phi^\star(m') \in G_j^{\mathbb{W}}$ with $j \leq l$. The analogous result holds for women.

Proof. Apply Proposition 5 to Proposition 6. □

Corollaries 3 and 4 show that, in any interior stable allocation, players with lower marginal costs exert higher efforts and, surprisingly, receive higher benefits than players with higher marginal costs. We saw both of these patterns in the interior stable allocation we found in Example 1.

Proposition 6 also allows us to examine the "rank-order" effects of a change in a player's type. For concreteness, suppose that (i) the type of man m increases from θ_m to θ_m', (ii) all players are partnered in every interior stable allocation, and (iii) each man has unique type and produces a unique benefit in every interior stable allocation.[25] Let $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ be an interior stable allocation before the increase, with benefits $\{b_i^\star\}_{i\in\mathcal{N}}$, and let $(\phi', \bar{\boldsymbol{x}}')$ be an interior stable allocation after the increase, with benefits $\{b_i'\}_{i\in\mathcal{N}}$. Since m's type increases, his rank-order type weakly increases, i.e., there are more men with lower types $|\{m'|\theta_{m'} \leq$

[25] Assumption 7 (below) is sufficient to ensure that all players are partnered and the conditions described in Footnote 22 are sufficient to ensure that each man produces a unique benefit when he has a unique type.

$\theta'_m\}| \geq |\{m'|\theta_{m'} \leq \theta_m\}|$. Thus, Proposition 6 gives that $|\{m'|b'_{m'} \leq b'_m\}| \geq |\{m'|b^\star_{m'} \leq b^\star_m\}|$, i.e., m produces a higher benefit than more men in $(\phi', \bar{\boldsymbol{x}}')$ than in $(\phi^\star, \bar{\boldsymbol{x}}^\star)$, so his rank-order benefit (and rank-order effort) increase. It follows from Proposition 5 that the rank-order benefit m receives also increase. Of course, this tells us nothing about whether the *levels* of benefits m produces and receives increase; we'll revisit this point in Proposition 9.

We prove Proposition 6 by formalizing the intuition discussed above.

Proof of Proposition 6. Let $\{z^\star_i\}_{i\in\mathcal{N}}$ be the associated efforts of $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ and $\{b^\star_i\}_{i\in\{j|\phi^\star(j)\neq j\}}$ be the associated benefits. Let m and m' be two partnered men with $\theta_m < \theta_{m'}$. Then, $m \in G^{\mathbb{M}}_l$ for some l and $m' \in G^{\mathbb{M}}_j$ for some j. We need to establish that $j \leq l$. If $l = J_{\mathbb{M}}$, then this is trivially true, so we take $l < J_{\mathbb{M}}$. We argue by contradiction. Suppose that $l < j$, i.e., $b^\star_m > b^\star_{m'}$, which implies $z^\star_m > z^\star_{m'}$. Let $w = \phi^\star(m)$ and $w' = \phi^\star(m')$. Proposition 5 gives that $b^\star_w > b^\star_{w'}$, which implies $z^\star_w > z^\star_{w'}$. We'll show that m' and w block $(\phi^\star, \bar{\boldsymbol{x}}^\star)$. But first, we need two preliminary facts.

Fact One: m does weakly better in $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ than he does by matching with w' at agreement $(z^\star_{m'}, z^\star_{w'})$, i.e., $b^\star_w - b^\star_{w'} \geq c(\theta_m, z^\star_m) - c(\theta_m, z^\star_{m'})$. Consider a possible match of m and w' with agreement $(x_1, z^\star_{w'})$, where $x_1 \in [0, \beta]$. For $x_1 > z^\star_{m'}$, we have that $u_{w'}(m, x_1, z^\star_{w'}) > u_{w'}(\phi^\star, \bar{\boldsymbol{x}}^\star)$ since $b(x_1) > b^\star_{m'} = b(z^\star_{m'})$. Thus, the stability of $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ implies, via the no blocking requirement, that $u_m(\phi^\star, \bar{\boldsymbol{x}}^\star) \geq u_m(w', x_1, z^\star_{w'})$. Simplifying gives that $b^\star_w - b^\star_{w'} \geq c(\theta_m, z^\star_m) - c(\theta_m, x_1)$ for $x_1 > z^\star_{m'}$. The desired result follows from the continuity of the cost function in effort.

Fact Two: m' does strictly better by matching with w at agreement $(z^\star_m, z^\star_w)$ than he does in $(\phi^\star, \bar{\boldsymbol{x}}^\star)$, i.e., $b^\star_w - b^\star_{w'} > c(\theta_{m'}, z^\star_m) - c(\theta_{m'}, z^\star_{m'})$. Since $z^\star_{m'} < z^\star_m$, Assumption 5 gives that $c(\theta_{m'}, z^\star_m) - c(\theta_{m'}, z^\star_{m'}) < c(\theta_m, z^\star_m) - c(\theta_m, z^\star_{m'})$. The desired result now follows from Fact One.

Now we show that m' and w block $(\phi^\star, \bar{\boldsymbol{x}}^\star)$. They block with agreement $(x_1, z^\star_w)$ if there is an $x_1 \in [0, \beta]$ with $u_w(m', x_1, z^\star_w) > u_w(\phi^\star, \bar{\boldsymbol{x}}^\star)$ and $u_{m'}(w, x_1, z^\star_w) > u_m(\phi^\star, \bar{\boldsymbol{x}}^\star)$, i.e., with (i) $b(x_1) > b(z^\star_m)$ and (ii) $b^\star_w - b^\star_{w'} > c(\theta_{m'}, x_1) - c(\theta_{m'}, z^\star_{m'})$. Consider $x_1 = z^\star_m + \delta$, where $\delta > 0$. Since $b(y)$ is strictly increasing, we have that $z^\star_m + \delta$ satisfies equation (i). Fact Two and the continuity of $c(\theta, y)$ in y imply that we can pick δ sufficiently small such that $z^\star_m + \delta$ satisfies equation (ii). Since $z^\star_m$ is interior, we can make δ small enough so that $z^\star_m + \delta \in [0, \beta]$. Thus, m' and w block $(\phi^\star, \bar{\boldsymbol{x}}^\star)$, a contradiction. □

Payoffs and Types

In this subsection, we develop our results on the relationship between types and payoffs. Corollary 4 tells us that higher types receive higher benefits from their matches. At the same time, Corollary 3 tells us higher types exert more effort, implying they may incur higher costs. Thus, it's unclear whether higher types actually do better than lower types.

To resolve this quandary, it suffices to make the following assumption.

Assumption 6. Common Cost of Minimum Effort.
There is a common cost of minimum effort, i.e., $c(\theta, 0) = d$ for all types θ, where $d \geq 0$.[26]

An immediate implication of this assumption and Assumption 5 is the following.

Lemma 8. Higher Types Have Lower Costs.
When Assumptions 5 and 6 hold, cost is strictly decreasing in type, i.e., $c(\theta, y) > c(\theta', y)$ when $\theta < \theta'$ and $y > 0$.

Proof. Obvious and omitted. □

This lemma plays a key role in the next proposition, which examines the relationship between type and payoff.

Proposition 7. Payoff and Type.
Let Assumptions 5 and 6 hold, and let $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ be an interior stable allocation. Then, higher type men obtain higher payoffs. In particular, for two men m and m', $\theta_m \leq \theta_{m'}$ implies that $u_m(\phi^\star, \bar{\boldsymbol{x}}^\star) \leq u_{m'}(\phi^\star, \bar{\boldsymbol{x}}^\star)$, and $\theta_m < \theta_{m'}$ implies that $u_m(\phi^\star, \bar{\boldsymbol{x}}^\star) < u_{m'}(\phi^\star, \bar{\boldsymbol{x}}^\star)$ when both men are partnered. The analogous result holds for women.

This result (i) applies to any interior stable allocation and (ii) gives that men with the same type have the same payoff. The next two examples illustrate the proposition.

Example 1 (Continued). Higher Types have Higher Payoffs.
Since Assumption 6 holds, the Proposition 7 gives that strictly higher types earn strictly more. Indeed, we see this in the stable allocation we found. The high types, man 1 and woman 4, each earn $3/8$ $(= 1 - 1/8 - 1/2 \cdot 1^2)$, while the low types, man 2 and woman 3, each earn $1/8$ $(= 1/2 - 1/8 - (1/2)^2)$. △

Example 4 (Continued). Equal Types have Equal Payoffs.
Since Assumption 6 holds, the Proposition 7 gives that both men earn the same. In fact, man 1 earns $1/2$ $(= 1 - 1/2)$ and man 2 also earns $1/2$ $(= \frac{1}{2}(1 + q^2) - \frac{1}{2}q^2$, where $q \approx 0.733)$ in the stable allocation we found. △

The intuition for Proposition 7 is that higher types can "imitate" and outcompete weakly lower types whenever they do strictly better. To elaborate, suppose there is an interior stable allocation where a lower type man m makes *strictly* more than a (weakly) higher type man m'. Since m' has a lower cost, he earns (weakly) more than m when he receives same benefit as m and exerts the same effort as m. Thus, m' can do strictly better by imitating and outcompeting m, i.e., by offering w, the partner of m, a slightly higher benefit than

[26] If Assumption 4 holds, then d must be greater than $b(0)$.

she's currently receiving. In doing so, he obtains the same benefit as m and incurs (at most) a slightly higher cost, so his payoff increases. Hence, m' and w block. It follows that a necessary condition of stability is that higher types make at least as much as lower types. The proof formalizes this intuition.

Since the intuition (and thus the proof) only require that m' has a lower cost than m, the proposition also holds when (i) higher types have lower fixed costs or (ii) men and women have different benefit functions. The proposition also obtains when the benefit function is increasing type and effort since this does not diminish m's ability (or desire) to imitate and outcompete m.

As the intuition suggests, the continuity of effort is essential for Proposition 7. When efforts are discrete, lower types may have *strictly* higher payoffs. For example, when $\mathcal{M} = \{1, 2\}$, $\mathcal{W} = \{3\}$, $b(y) = y$, $c(\theta, y) = {}^{y}/_{\theta}$, $\theta_1 = \theta_3 = 2$ and $\theta_2 = 3/2$, and the set of feasible agreements is $\{0, {}^{1}/_{2}\}^2$, it's stable for man 2 to match with woman 3 with agreement $({}^{1}/_{2}, {}^{1}/_{2})$, while man 1 is single with an arbitrary agreement. The lower type man earns ${}^{1}/_{6}$ while the higher type man earns 0. In addition, equal types may have unequal earnings – e.g., the previous allocation is still stable when $\theta_2 = 2$.

We prove Proposition 7 in two steps. First, we give a lemma that establishes higher types are partnered whenever lower types are partnered. Subsequently, we use this lemma and the argument we sketched above to prove the proposition.

Lemma 9. Higher Types are Partnered when Lower Types are Partnered.
Let Assumptions 5 and 6 hold, and $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ be an interior stable allocation. Consider two men m and m' with $\theta_m < \theta_{m'}$. If m is partnered, then so too is m', i.e., $\phi^\star(m) \neq m \implies \phi^\star(m') \neq m'$. The analogous result holds for women.

Proof. See the Appendix. □

The intuition for the lemma is familiar: if m is partnered and m' isn't, then m' does strictly better by imitating and outcompeting m; see the proof for details.

Proof of Proposition 7. Let $\{z_i^\star\}_{i\in\mathcal{N}}$ be the associated efforts of $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ and $\{b_i^\star\}_{i\in\{j|\phi^\star(j)\neq j\}}$ be the associated benefits. Let m and m' be two different men. We first establish that $\theta_m < \theta_{m'}$ implies that $u_m(\phi^\star, \bar{\boldsymbol{x}}^\star) \leq u_{m'}(\phi^\star, \bar{\boldsymbol{x}}^\star)$, with strict inequality if both men are partnered. Subsequently, we establish that $\theta_m = \theta_{m'}$ implies that $u_m(\phi^\star, \bar{\boldsymbol{x}}^\star) = u_{m'}(\phi^\star, \bar{\boldsymbol{x}}^\star)$. The proposition follows.

Let $\theta_m < \theta_{m'}$. There are four cases to consider: (i) m and m' are both single, (ii) m is single and m' is partnered, (iii) m is partnered and m' is single, and (iv) both m and m' are partnered. If case (i), then both m and m' earn zero. If case (ii), then m earns zero and m' earns at least zero as $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ is individually rational. Since Lemma 9 gives that case (iii) is impossible, only case (iv) remains.

Consider case (iv). We need to establish that $u_m(\phi^\star, \bar{\boldsymbol{x}}^\star) < u_{m'}(\phi^\star, \bar{\boldsymbol{x}}^\star)$. We argue by contradiction. Suppose that $u_m(\phi^\star, \bar{\boldsymbol{x}}^\star) \geq u_{m'}(\phi^\star, \bar{\boldsymbol{x}}^\star)$, and let $w = \phi^\star(m)$ and $w' = \phi^\star(m')$. We'll show that m' and w block.

Man m' and woman w block with agreement $(x_1, z_w^\star)$ if there is an $x_1 \in [0, \beta]$ such that $u_w(m', x_1, z_w^\star) > u_w(\phi^\star, \bar{\boldsymbol{x}}^\star)$ and $u_{m'}(w, x_1, z_w^\star) > u_{m'}(\phi^\star, \bar{\boldsymbol{x}}^\star)$, i.e., such that (i) $b(x_1) > b(z_m^\star)$ and (ii) $b_w^\star - c(\theta_{m'}, x_1) > u_{m'}(\phi^\star, \bar{\boldsymbol{x}}^\star)$. Consider $x_1 = z_m^\star + \delta$, with $\delta > 0$. Since $b(y)$ strictly increasing, we have that $z_m^\star + \delta$ satisfies equation (i). Lemma 8 gives that $b_w^\star - c(\theta_{m'}, z_m^\star) > b_w^\star - c(\theta_m, z_m^\star) = u_m(\phi^\star, \bar{\boldsymbol{x}}^\star) \geq u_{m'}(\phi^\star, \bar{\boldsymbol{x}}^\star)$. Thus, the continuity of $c(\theta, y)$ in y implies that we may pick δ sufficiently small such that $z_m^\star + \delta$ satisfies equation (ii). Since $z_m^\star$ is interior, we may make δ small enough so that $z_m^\star + \delta \in [0, \beta]$. Thus, m' and w block $(\phi^\star, \bar{\boldsymbol{x}}^\star)$, a contradiction.

We now establish that $\theta_m = \theta_{m'}$ implies $u_m(\phi^\star, \bar{\boldsymbol{x}}^\star) = u_{m'}(\phi^\star, \bar{\boldsymbol{x}}^\star)$. We again argue via contradiction. Suppose (without loss) that $u_m(\phi^\star, \bar{\boldsymbol{x}}^\star) > u_{m'}(\phi^\star, \bar{\boldsymbol{x}}^\star)$. As above, m' and w block with agreement $(x_1, z_w^\star)$ if there is an $x_1 \in [0, \beta]$ with $u_w(m', x_1, z_w^\star) > u_w(\phi^\star, \bar{\boldsymbol{x}}^\star)$ and $u_{m'}(w, x_1, z_w^\star) > u_{m'}(\phi^\star, \bar{\boldsymbol{x}}^\star)$, i.e., with (i) $b(x_1) > b(z_m^\star)$ and (ii) $b_w^\star - b_{w'}^\star > c(\theta_{m'}, x_1) - c(\theta_{m'}, z_{m'}^\star)$. Since $\theta_m = \theta_{m'}$ and since $u_m(\phi^\star, \bar{\boldsymbol{x}}^\star) = b_w^\star - c(\theta_m, z_m^\star) > u_{m'}(\phi^\star, \bar{\boldsymbol{x}}^\star) = b_{w'}^\star - c(\theta_{m'}, z_{m'}^\star)$, we have $b_w^\star - b_{w'}^\star > c(\theta_{m'}, z_m^\star) - c(\theta_{m'}, z_{m'}^\star)$. Thus, at $x_1 = z_m^\star + \delta$, where $\delta > 0$, we have that both equations (ii) and (ii) are true for δ sufficiently small since $b(y)$ is strictly increasing and $c(\theta, y)$ is continuous in y. Since $z_m^\star$ is interior, we may take δ sufficiently small such that $z_m^\star + \delta \in [0, \beta]$. Thus, m' and w block, a contradiction. $\square$

Remark. As is clear from the proof, the result that men of the same type earn the same payoff is independent of Assumptions 5 and 6.

Assortative Matching in Endowed Types

In this subsection, we show that, as Corollary 4 suggests, there's at least one stable allocation where higher type men match with higher type women. We follow Legros and Newman [28] and say that an allocation $(\phi, \bar{\boldsymbol{x}})$ **exhibits assortative matching in types** if, for any two partnered men m and m' with $\theta_m < \theta_{m'}$, we have that m' matches with a higher type woman than m does, i.e., $\theta_w \leq \theta_{w'}$, where $w = \phi(m)$ and $w' = \phi(m')$.[27]

Proposition 8. Assortative Matching in Types.
Let Assumptions 4 and 5 hold, then there is an interior stable allocation $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ that exhibits assortative matching in types.

Not all interior stable allocations exhibit assortative matching because players may match

[27]Defining assortative matching is terms of men is without loss. If $(\phi, \bar{\boldsymbol{x}})$ exhibits assortative matching and if $\theta_w < \theta_{w'}$ for two women w and w', then it's readily verified that $\theta_{\phi^\star(w)} \leq \theta_{\phi^\star(w')}$.

in arbitrary ways within their benefit groups. For instance, if there are two men m and m' in $G_l^{\mathbb{M}}$, with $\theta_m < \theta_{m'}$, and two women w and w' in $G_l^{\mathbb{W}}$, with $\theta_w < \theta_{w'}$, it's possible that m is matched to w' and m' is matched to w.[28] That said, all players in the same benefit group *receive* the same benefit. Hence, we can rematch these men and women so that higher types are matched to higher types while preserving players' payoffs and, thus, the stability of the allocation. Accordingly, we obtain Proposition 8.

We prove Proposition 8 by formally developing this rematching procedure and then showing that it functions as desired by applying Propositions 5 and 6. In the process, we'll see that it only makes use of the additive nature of payoffs. Hence, it continues to function (and Proposition 8 obtains) when (i) men and women have different benefit functions or (ii) the benefit function is increasing in type and effort.

Proof of Proposition 8. By Corollary 1, there is an interior stable allocation $(\phi', \bar{\boldsymbol{x}}')$. We proceed in three steps. First, we describe the rematching procedure that constructs a new allocation $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ from $(\phi', \bar{\boldsymbol{x}}')$. Second, we establish that $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ is an interior stable allocation. Third, we show that $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ exhibits assortative matching in types. Throughout, let $G_1^{\mathbb{M}}, \ldots, G_{J_{\mathbb{M}}}^{\mathbb{M}}$ and $G_1^{\mathbb{W}}, \ldots, G_{J_{\mathbb{M}}}^{\mathbb{W}}$ be the benefit groups associated with $(\phi', \bar{\boldsymbol{x}}')$, and let z_i' and b_i' denote player i's effort and benefit in $(\phi', \bar{\boldsymbol{x}}')$. Also, let $z_i^\star$ and $b_i^\star$ denote player i's effort and benefit in $(\phi^\star, \bar{\boldsymbol{x}}^\star)$.

We first construct $(\phi^\star, \bar{\boldsymbol{x}}^\star) = (\phi^\star, \boldsymbol{x}^{1^\star}, \ldots, \boldsymbol{x}^{N^\star})$ from $(\phi', \bar{\boldsymbol{x}}') = (\phi', \boldsymbol{x}^{1'}, \ldots, \boldsymbol{x}^{N})$ as follows.

1. Set $\phi^\star$ so that single players remain single and partnered players are matched in descending order of their types in their benefit groups. For each $i \in \mathcal{N}$ with $\phi'(i) = i$, set $\phi^\star(i) = i$. For each $l \in \{1, \ldots, J_{\mathbb{M}}\}$:
   (a) List the men in $G_l^{\mathbb{M}}$ in descending order of their types (breaking ties randomly) and label them m_1, m_2, ..., $m_{|G_l^{\mathbb{M}}|}$. So m_1 is the highest type man in $G_l^{\mathbb{M}}$, m_2 is the second highest type man, and so on. Likewise, list the women in $G_l^{\mathbb{W}}$ in descending order (breaking ties randomly) of their types and label them w_1, ..., $w_{|G_l^{\mathbb{M}}|}$. (Recall that $|G_l^{\mathbb{M}}| = |G_l^{\mathbb{W}}|$ by Proposition 5.)
   (b) Set $\phi^\star$ such that the j-th man and j-th woman on each list are matched, i.e., such that $\phi^\star(m_j) = w_j$ and $\phi^\star(w_j) = m_j$ for $j \in \{1, \ldots, |G_l^{\mathbb{M}}|\}$.
2. Set $\bar{\boldsymbol{x}}^\star$ so that players exert the same efforts. If $\phi^\star(i) = i$, then set $\boldsymbol{x}^{i^\star} = (z_i', 0)$ when i is a man and $\boldsymbol{x}^{i^\star} = (0, z_i')$ when i is a woman. If $\phi^\star(i) \neq i$, then set $\boldsymbol{x}^{i^\star} = (z_i', z_{\phi^\star(i)}')$ when i is a man and $\boldsymbol{x}^{i^\star} = (z_{\phi^\star(i)}', z_i')$ when i is a woman. (Thus, $z_i^\star = z_i'$ for all $i \in \mathcal{N}$ as desired.)

[28]If, however, the conditions of Footnote 21 hold, then each benefit group contains only a single kind of player. Thus, *every* interior stable allocation exhibits assortative matching in types. We develop this result in the Supplement.

This concludes the construction. Notice (i) that the same players are partnered under ϕ' and $\phi^\star$ by step (1) and (ii) that $\boldsymbol{x}^{i^\star} = \boldsymbol{x}^{\phi^\star(i)^\star}$ for each player i by step (2).

Second, we establish that $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ is an interior stable allocation. Since $\boldsymbol{x}^{i^\star} = \boldsymbol{x}^{\phi^\star(i)^\star}$ for each player i, $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ is an allocation. It's also interior because z_i' is interior for each partnered player i. Hence, we only need establish that $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ is stable. To these ends, observe that $u_i(\phi^\star, \bar{\boldsymbol{x}}^\star) = u_i(\phi', \bar{\boldsymbol{x}}')$ for all $i \in \mathcal{N}$. (This is trivial if i is single, so we take i to be partnered. By step (2), i's effort is unchanged, so his or her costs are unchanged. Since $\phi^\star(i)$ and $\phi'(i)$ are both in the same benefit group in $(\phi', \bar{\boldsymbol{x}}')$ by step (1) and since we keep their efforts constant by step (2), the benefit i receives is unchanged. Thus, i earns the same payoff in $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ and $(\phi', \bar{\boldsymbol{x}}')$.) It follows that $(\phi^\star, \bar{\boldsymbol{x}}^\star) \in P(\phi', \bar{\boldsymbol{x}}')$, so Lemma 7 implies that $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ is stable.

Third, we establish that $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ exhibits assortative matching in types. Let m and m' be partnered men such that $\theta_m < \theta_{m'}$. Proposition 6 gives that there are two cases: (i) $b^\star_m = b^\star_{m'}$ or (ii) $b^\star_m < b^\star_{m'}$. Let $w = \phi^\star(m)$ and $w' = \phi^\star(m')$. If case (i), then in step (1), m' occupies an earlier position in the list than m. Thus, m' is matched to a higher type woman than m, i.e., $\theta_w \leq \theta_{w'}$. If case (ii), we establish that $\theta_w \leq \theta_{w'}$ via contradiction. Suppose $\theta_w > \theta_{w'}$, then Proposition 6 gives that $b^\star_w \geq b^\star_{w'}$. Yet, $b^\star_m < b^\star_{m'}$ implies that $b^\star_w < b^\star_{w'}$ by Proposition 5, a contradiction. $\square$

Remark. It's clear from the proof that every interior stable allocation is payoff equivalent to an interior stable allocation that exhibits assortative matching in types. Thus, Proposition 3 of Legros and Newman [28] implies that the Pareto frontier of each couple satisfies generalized increasing differences, which is their sufficient condition on the Pareto frontier for assortative matching in types. In light of this, one can think of our result as giving natural conditions on the primitive payoffs that imply Legros and Newman's general sufficient condition.

Comparative Statics of Benefits, Efforts, and Payoffs

In this subsection, we first make the idea of a "symmetric stable allocation" precise, and we prove several properties of this allocation (e.g., its existence and uniqueness). Then we examine how increases in players' types affect their efforts, benefits, and payoffs in this allocation. We make the following assumption.

Assumption 7. Universal Strict Agreeability and Universal Intolerability.
There are an equal number of men and women, i.e., $M = N/2$. For all θ and θ' in Θ:
(i) $b(x_2) - c(\theta, x_1) > 0$ and $b(x_1) - c(\theta', x_2) > 0$ for some $(x_1, x_2) \in X$.
(ii) $c(\theta, \beta) > b(\beta)$ and $c(\theta, y) > b(0)$ for all $y \in [0, \beta]$.

Part (i) guarantees any type of man and any type of woman can find an agreement that makes them strictly better off than if they're single. Part (ii) guarantees that intolerability holds. Thus, Assumption 7 implies Assumption 4.

Let $\{\theta_i\}_{i\in\mathcal{N}}$ denote the endowment of players' types.

Definition. We say that the type endowment $\{\theta_i\}_{i\in\mathcal{N}}$ is *symmetric* when:
(i) No two men have the same type and no two women have the same type, i.e., $\theta_m \neq \theta_{m'}$ for any two men m and m' and $\theta_w \neq \theta_w'$ for any two women w and w'.
(ii) Men and women are endowed with the same types, i.e., $\cup_{i\in\mathcal{M}}\{\theta_i\} = \cup_{i\in\mathcal{W}}\{\theta_i\}$.

For instance, the type endowment in Example 1 is symmetric: the endowment is $\{\theta_1, \theta_2, \theta_3, \theta_4\} = \{2, 1, 1, 2\}$, and we have $\{\theta_i\}_{i\in\mathcal{M}} = \{\theta_1, \theta_2\} = \{2, 1\}$ and $\{\theta_i\}_{i\in\mathcal{W}} = \{\theta_3, \theta_4\} = \{1, 2\}$.

Next, we describe the construction of a "symmetric stable allocation."

Construction. Symmetric Stable Allocation.
Let Assumption 7 hold and let the type endowment $\{\theta_i\}_{i\in\mathcal{N}}$ be symmetric. Label the men m_1, m_2, ..., m_M in descending order of their types; so, m_1 is the highest type man, m_2 is the second highest type man, and so on. Likewise, label the women w_1, w_2, ..., w_M in descending order of their types.

We construct a **symmetric stable allocation** $(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger) = (\phi^\dagger, \boldsymbol{x}^{1^\dagger}, \ldots, \boldsymbol{x}^{N^\dagger})$ as follows. Set $\phi^\dagger$ such that men and women with the same rank-order type are matched, i.e., such that $\phi^\dagger(m_l) = w_l$ and $\phi^\dagger(w_l) = m_l$ for each $l \in \{1, \ldots, M\}$. (Thus, every player is partnered.) Set $\bar{\boldsymbol{x}}^\dagger$ such that, for each $l \in \{1, \ldots, M\}$, $\boldsymbol{x}^{m_l^\dagger} = \boldsymbol{x}^{w_l^\dagger} = (x_l^\dagger, x_l^\dagger)$ where

$$x_l^\dagger = \max\{\arg\max_{y\in[0,\beta]} b(y) - c(\theta_{m_l}, y)\}, \tag{5.1}$$

i.e., m_l and w_l exert the same effort and this effort is the largest solution of $\max_{y\in[0,\beta]} b(y) - c(\theta_{m_l}, y)$. ○

One may verify, for instance, that the stable allocation given in Example 1 is actually a symmetric stable allocation.

The next lemma gives a few properties of the symmetric stable allocation. It shows, for instance, that the symmetric stable allocation is actually stable. Also, we say an allocation $(\phi', \bar{\boldsymbol{x}}')$ is **welfare maximizing** if it maximizes $T(\phi, \bar{\boldsymbol{x}})$ on the set of feasible allocations.[29]

Lemma 10. Properties of the Symmetric Stable Allocation.
Let Assumption 7 hold and let the endowment of types $\{\theta_i\}_{i\in\mathcal{N}}$ be symmetric. Then (i) a symmetric stable allocation $(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger)$ exists and is unique (i.e., the procedure described above only produces one allocation), (ii) $(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger)$ is an interior Pareto stable allocation, and (iii)

[29] While welfare maximizing allocations always exist in the Effort Game, they generally aren't stable; see the Supplement for details.

$(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger)$ *is welfare maximizing. In addition, for each player i, we have $\theta_i = \theta_{\phi^\dagger(i)}$ and*

$$u_i(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger) = \max_{y \in [0,\beta]} b(y) - c(\theta_i, y). \tag{5.2}$$

Proof. See the Appendix. □

In light of Lemma 10, we view the symmetric stable allocation as focal: it maximizes welfare and it treats equals equally in that matched players have the same type, exert the same effort, and earn the same payoffs. Hence, we develop comparative statics for it in the next proposition. We write $u_i(\phi, \bar{\boldsymbol{x}}, \theta_i)$ to emphasize the dependence of i's payoff in the allocation $(\phi, \bar{\boldsymbol{x}})$ on her type θ_i.

Proposition 9. Comparative Statics of the Symmetric Stable Allocation.
Let Assumptions 5, 6, and 7 hold. Let $\{\theta_i\}_{i\in\mathcal{N}}$ and $\{\theta'_i\}_{i\in\mathcal{N}}$ be symmetric endowments of types such that $\theta'_i \geq \theta_i$ for each player i. Let $(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger)$ be the symmetric stable allocation when types are $\{\theta_i\}_{i\in\mathcal{N}}$, let $(\phi', \bar{\boldsymbol{x}}')$ be the symmetric stable allocation when types are $\{\theta'_i\}_{i\in\mathcal{N}}$, and let $\{z^\dagger_i\}_{i\in\mathcal{N}}$ and $\{z'_i\}_{i\in\mathcal{N}}$ be players' efforts in $(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger)$ and $(\phi', \bar{\boldsymbol{x}}')$ respectively. Then, for each player i, as types increase from $\{\theta_i\}_{i\in\mathcal{N}}$ to $\{\theta'_i\}_{i\in\mathcal{N}}$,
(i) The effort i exerts increases, i.e., $z^\dagger_i \leq z'_i$.
(ii) The effort exerted by i's match increases, i.e., $z^\dagger_{\phi^\dagger(i)} \leq z'_{\phi'(i)}$.
(iii) The payoff of i increases, i.e., $u_i(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger, \theta_i) \leq u_i(\phi', \bar{\boldsymbol{x}}', \theta'_i)$.

Since $b(y)$ is strictly increasing, (i) implies that the benefit player i produces increases and (ii) implies that the benefit i receives from his or her match increases. Thus, if a player's opportunity cost of effort decreases, then his or her effort, benefits produced and received, and payoff increase. The next example illustrates these results.

Example 5. Effort and Welfare Comparative Statics for a Variant of Example 1.

Let $\mathcal{M} = \{1, 2\}$ and $\mathcal{W} = \{3, 4\}$. Let $X = [0, 2]^2$. Let $\Theta = \{1, 2, 2.1\}$. Let $b(y) = y$ and $c(\theta, y) = \frac{1}{\theta}y^2 + 1/8$. It's readily verified that Assumptions 5, 6, and 7 hold for any assignment of types to players. Let $\theta_1 = \theta_4 = 2$ and $\theta_2 = \theta_3 = 1$ be the initial endowment of types. And let $\theta'_1 = \theta'_3 = 2$ and $\theta'_2 = \theta'_4 = \frac{21}{10}$ be the new endowment of types. Since both endowments are trivially symmetric and since $\theta'_i \geq \theta_i$ for each player i, Proposition 9 gives that all players exert more effort, produce and receive greater benefits, and are better off under the new endowment.

We can see this directly. The symmetric stable allocation under $\{\theta_i\}_{i\in\mathcal{N}}$ is $(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger) = (\phi^\dagger, \boldsymbol{x}^{1^\dagger}, \boldsymbol{x}^{2^\dagger}, \boldsymbol{x}^{3^\dagger}, \boldsymbol{x}^{4^\dagger})$, where $\phi^\dagger(1) = 4$ and $\boldsymbol{x}^{1^\dagger} = \boldsymbol{x}^{4^\dagger} = (1, 1)$, and $\phi^\dagger(2) = 3$ and $\boldsymbol{x}^{2^\dagger} = \boldsymbol{x}^{3^\dagger} = (1/2, 1/2)$. In this allocation, 1 and 4 each earn $3/8 = 0.375$, while 2 and 3 each earn $1/8 = 0.125$. And the symmetric stable allocation under $\{\theta'_i\}_{i\in\mathcal{N}}$ is $(\phi', \bar{\boldsymbol{x}}') = (\phi', \boldsymbol{x}^{1'}, \boldsymbol{x}^{2'}, \boldsymbol{x}^{3'}, \boldsymbol{x}^{4'})$, where $\phi'(1) = 3$ and $\boldsymbol{x}^{1'} = \boldsymbol{x}^{3'} = (1, 1)$, and $\phi'(2) = 4$ and $\boldsymbol{x}^{2'} = \boldsymbol{x}^{4'} = (21/20, 21/20)$. In this allocation, 1 and 3 each earn $3/8 = 0.375$, while 2 and 4 each earn $2/5 = 0.4$. The players efforts, benefits, and payoffs are clearly higher in $(\phi', \bar{\boldsymbol{x}}')$ than $(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger)$. $\triangle$

The intuition for Proposition 9 is that cost is submodular and strictly decreasing in type. Thus, as types increase, equation (5.1) ensures that players' efforts increase and equation (5.2) ensures that their payoffs increase. The proof below formalizes this intuition. Interestingly, Assumption 7 and the symmetric endowment of types are essential for Proposition 9. Without them, an increase in types may make a player strictly worse off even in every interior stable allocation; see the Supplement for details.

Proof of Proposition 9. We need a preliminary fact: $h(\theta) = \max\{\arg\max_{y \in [0,\beta]} b(y) - c(\theta, y)\}$ is non-decreasing. Since $-c(\theta, y)$ is supermodular by Assumption 5 this follows directly from Topkis' Monotonicity Theorem (Theorem 2.8.1 of [43]).

Consider a woman w whose type increases from θ_w to θ'_w. In $(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger)$, she and her match $\phi^\dagger(w)$ both exert effort $h(\theta_w)$ by construction, while in $(\phi', \bar{\boldsymbol{x}}')$, she and her match $\phi'(w)$ both exert effort $h(\theta'_w)$. Since $h(\theta_w) \leq h(\theta'_w)$, we have that w's effort increases and the effort of her match increases, even though the identity of her match may change. Since the analogous argument holds for each man, parts (i) and (ii) of the proposition follow.

It remains to show that $u_i(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger, \theta_i) \leq u_i(\phi', \bar{\boldsymbol{x}}', \theta'_i)$ for each player i. Write $u_i(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger, \theta_i) = \max_{y \in [0,\beta]} b(y) - c(\theta_i, y) \leq \max_{y \in [0,\beta]} b(y) - c(\theta'_i, y) = u_i(\phi', \bar{\boldsymbol{x}}', \theta'_i)$, where the equalities are due to Lemma 10 and the weak inequality is due to Lemma 8. $\square$

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A Appendix: Omitted Proofs

Proof of Lemma 3. We take $F(\phi)$ to be nonempty because the empty set is trivially compact. We have that $F(\phi)$ is bounded as X^N is compact. We also have that $F(\phi)$ is closed because (i) X^N and $A(\phi)$ are closed and (ii) the function $u_i(\phi, \bar{\boldsymbol{x}})$ is continuous in $\bar{\boldsymbol{x}}$ for each $i \in \mathcal{N}$ per Lemma 2. □

Proof of Lemma 4. We take $D_\phi(c)$ to be nonempty since the empty set is trivially open. Let $(m, w, \boldsymbol{x}) = c$. Since $u_m(\phi, \bar{\boldsymbol{x}})$ and $u_w(\phi, \bar{\boldsymbol{x}})$ are continuous in $\bar{\boldsymbol{x}}$ per Lemma 2, we have that $\{\bar{\boldsymbol{x}} \in \mathbb{R}^{kN} | u_m(\phi, \bar{\boldsymbol{x}}) < u_m(w, \boldsymbol{x})\}$ and $\{\bar{\boldsymbol{x}} \in \mathbb{R}^{kN} | u_w(\phi, \bar{\boldsymbol{x}}) < u_w(m, \boldsymbol{x})\}$ are open sets. Since $D_\phi(c) = \{\bar{\boldsymbol{x}} \in \mathbb{R}^{kN} | u_m(\phi, \bar{\boldsymbol{x}}) < u_m(w, \boldsymbol{x})\} \cap \{\bar{\boldsymbol{x}} \in \mathbb{R}^{kN} | u_w(\phi, \bar{\boldsymbol{x}}) < u_w(m, \boldsymbol{x})\}$, it follows that $D_\phi(c)$ is open. □

Proof of Lemma 5. It's easily seen that $\mathcal{S}(\phi) = F(\phi) \cap \overline{D(\phi)}$, where $D(\phi) = \cup_{c \in C} D_\phi(c)$ is the set of agreement vectors such that the pair $(\phi, \bar{\boldsymbol{x}})$ is blocked by some man and woman for each $\bar{\boldsymbol{x}} \in D_\phi$ and where $\overline{D(\phi)}$ is the complement of $D(\phi)$. Since $F(\phi)$ is closed (per Lemma 3) and $\overline{D(\phi)}$ is closed (since $D(\phi)$ is open because it's the union of open sets by Lemma 4), we have that $\mathcal{S}(\phi)$ is closed. Since $\mathcal{S}(\phi) \subset X^N$, it's also bounded. $\square$

Proof of Lemma 6. By Proposition 1, we have that $\mathcal{S}$ is nonempty. Thus, $\Phi_{\mathcal{S}}$ is nonempty (and finite), as is $\mathcal{S}(\phi)$ for each $\phi \in \Phi_{\mathcal{S}}$. For each $\phi \in \Phi_{\mathcal{S}}$, we have the existence of an $\bar{\boldsymbol{x}}_\phi \in \mathcal{S}(\phi)$ that solves $\max_{\bar{\boldsymbol{x}} \in \mathcal{S}(\phi)} T(\phi, \bar{\boldsymbol{x}})$ by the Extreme Value Theorem applies since $T(\phi, \bar{\boldsymbol{x}})$ is continuous in $\bar{\boldsymbol{x}}$ by Lemma 2, $\mathcal{S}(\phi)$ is nonempty, and $\mathcal{S}(\phi)$ is compact by Lemma 5. Let $I = \cup_{\phi \in \Phi_{\mathcal{S}}} \{(\phi, \bar{\boldsymbol{x}}_\phi)\}$ and let $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ solve $\max_{(\phi, \bar{\boldsymbol{x}}) \in I} T(\phi, \bar{\boldsymbol{x}})$. A solution exists since I is finite.

We now establish that $T(\phi^\star, \bar{\boldsymbol{x}}^\star) \geq T(\phi, \bar{\boldsymbol{x}})$ for all $(\phi, \bar{\boldsymbol{x}}) \in \mathcal{S}$. Suppose not, then there is a $(\phi', \bar{\boldsymbol{x}}') \in \mathcal{S}$ with $T(\phi', \bar{\boldsymbol{x}}') > T(\phi^\star, \bar{\boldsymbol{x}}^\star)$. Thus, $\mathcal{S}(\phi') \neq \emptyset$ and $(\phi', \bar{\boldsymbol{x}}_{\phi'}) \in I$. Since, $T(\phi', \bar{\boldsymbol{x}}_{\phi'}) \leq T(\phi^\star, \bar{\boldsymbol{x}}^\star)$, we have $T(\phi', \bar{\boldsymbol{x}}_{\phi'}) < T(\phi', \bar{\boldsymbol{x}}')$, a contradiction of the optimality of $\bar{\boldsymbol{x}}_{\phi'}$ on $\mathcal{S}(\phi')$. $\square$

Remark. It's clear from the proof that there is a stable allocation that maximizes the total payoff of any group of players (e.g., the total payoff of women).

Proof of Lemma 7. These results are almost obvious. We first establish $L7.1$. Let $(\phi^\star, \bar{\boldsymbol{x}}^\star) \in P(\phi', \bar{\boldsymbol{x}}')$. By construction of $P(\cdot)$, $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ is feasible and individually rational, so we only need to show that it can't be blocked. We do this by contradiction. Suppose a man m and a woman w block $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ with agreement $\boldsymbol{x} \in X$, then $u_m(w, \boldsymbol{x}) > u_m(\phi^\star, \bar{\boldsymbol{x}}^\star) \geq u_m(\phi', \bar{\boldsymbol{x}}')$ and $u_w(m, \boldsymbol{x}) > u_w(\phi^\star, \bar{\boldsymbol{x}}^\star) \geq u_w(\phi', \bar{\boldsymbol{x}}')$ where the weak inequality follows from the definition of $P(\phi^\star, \bar{\boldsymbol{x}}^\star)$. Thus, m and w also block $(\phi', \bar{\boldsymbol{x}}')$, a contradiction.

Now we establish $L7.2$. Since $(\phi^\star, \bar{\boldsymbol{x}}^\star) \in P(\phi', \bar{\boldsymbol{x}}')$, we have $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ is stable by $L7.1$. We only need to show that there is no other feasible allocation $(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger)$ such that (i) $u_i(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger) \geq u_i(\phi^\star, \bar{\boldsymbol{x}}^\star)$ for all $i \in \mathcal{N}$ and (ii) $u_i(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger) > u_i(\phi^\star, \bar{\boldsymbol{x}}^\star)$ for some $i \in \mathcal{N}$. We do this by contradiction. Suppose there is such a $(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger)$, then $T(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger) > T(\phi^\star, \bar{\boldsymbol{x}}^\star)$. Thus, $(\phi^\star, \bar{\boldsymbol{x}}^\star)$ does not solve the maximization problem, a contradiction. $\square$

Proof of Lemma 9. We prove the lemma via contradiction. Suppose that man m is partnered, while man m' isn't. Let $w = \phi^\star(m)$. Let $b_m^\star$ and $z_m^\star$ be the benefit and effort of m in $(\phi^\star, \bar{\boldsymbol{x}}^\star)$, and let $b_w^\star$ and $z_w^\star$ be the benefit and effort of w in $(\phi^\star, \bar{\boldsymbol{x}}^\star)$. We'll establish that man m' and woman w block.

They block with agreement $(x_1, z_w^\star)$ if there is an $x_1 \in [0, \beta]$ with $u_w(m', x_1, z_w^\star) > u_w(\phi^\star, \bar{\boldsymbol{x}}^\star)$ and $u_{m'}(w, x_1, z_w^\star) > u_{m'}(\phi^\star, \bar{\boldsymbol{x}}^\star)$. Simplifying gives that m' and w block if there is an $x_1 \in [0, \beta]$ such that (i) $b(x_1) > b(z_m^\star)$ and (ii) $b_w^\star - c(\theta_{m'}, x_1) > 0$. Consider $x_1 = z_m^\star + \delta$, where $\delta > 0$. Since $b(y)$ strictly increasing, we have that $z_m^\star + \delta$ satisfies equation (i). Individual rationality and Lemma 8 give that $b_w^\star - c(\theta_{m'}, z_m^\star) > b_w^\star - c(\theta_m, z_m^\star) = u_m(\phi^\star, \bar{\boldsymbol{x}}^\star) \geq 0$. Hence, the continuity of cost in effort allows us to pick δ sufficiently small so that $z_m^\star + \delta$ satisfies equation (ii). Since $z_m^\star$ is interior, we may make δ small enough so that $z_m^\star + \delta \in [0, \beta]$. Thus, m' and w block $(\phi^\star, \bar{\boldsymbol{x}}^\star)$, a contradiction.□

Proof of Lemma 10. We prove the lemma in five steps. First, we argue existence and uniqueness of the symmetric stable allocation $(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger)$. Second, we establish that players' payoffs are given by equation (5.2). Third, we leverage this equation and Assumption 7 to show that the symmetric stable allocation is actually stable. Fourth, we show that the symmetric stable allocation is interior and is Pareto optimal. Fifth, we establish that the symmetric stable allocation maximizes welfare.

We first establish existence and uniqueness of $(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger)$. Since the benefit function and the cost function are continuous in y, $\arg\max_{y \in [0,\beta]} b(y) - c(\theta_{m_l}, y)$ is non-empty and compact by standard arguments. Thus, the it has a unique maximal element in the standard order, so $x_l^\dagger$ of equation (5.1) exists and is unique for each $l \in \{1, \ldots, M\}$. It follows that $(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger)$ exists and, since $\phi^\dagger$ is uniquely determined by the construction because no two players of the same gender have the same type, that $(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger)$ is unique. It's an allocation since $\phi^\dagger$ is a matching by construction and $\bar{\boldsymbol{x}}^\dagger \in A(\phi^\dagger)$ by construction.

Recall that the men and women are labeled $m_1, \ldots, m_M$ and $w_1, \ldots, w_M$ in descending order of their types. Since the type endowment is symmetric, we have $\theta_{m_1} = \theta_{w_1}$, $\theta_{m_2} = \theta_{w_2}$, …, and $\theta_{m_M} = \theta_{w_M}$. That is, man m_1 and woman w_1 have the same type, man m_2 and woman w_2 have the same type, and so on.

Second, we establish that equation (5.2) holds. Suppose that player i is a woman with label w_j, then $\phi^\dagger(i)$'s label is m_j by construction. Thus, the previous paragraph gives $\theta_i = \theta_{w_j} = \theta_{m_j} = \theta_{\phi^\dagger(i)}$. Hence, $u_i(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger) = b(x_j^\dagger) - c(\theta_i, x_j^\dagger) = b(x_j^\dagger) - c(\theta_{m_j}, x_j^\dagger) = \max_{y \in [0,\beta]} b(y) - c(\theta_{m_j}, y) = \max_{y \in [0,\beta]} b(y) - c(\theta_i, y)$, where the third equality is due to optimality. Since the analogous argument applies if i is a man, equation (5.2) holds.

Third, we establish that $(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger)$ is stable. By construction, $(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger)$ is feasible because (i) $\phi^\dagger$ is a matching and (ii) $\bar{\boldsymbol{x}}^\dagger \in X^N$ since $x_l^\dagger \in [0, \beta]$ for all $l \in \{1, \ldots, M\}$. It's also individually rational. To see this, consider man m. Suppose that m's label is m_j, then m is matched to woman w_j and $\theta_m = \theta_{m_j} = \theta_{w_j}$. Consider the maximum of the sum of m and w_j's payoffs,

$$\max_{(x_1,x_2) \in X} b(x_2) - c(\theta_{m_j}, x_1) + b(x_1) - c(\theta_{w_j}, x_2)$$
$$= \max_{x_1 \in [0,\beta]} b(x_1) - c(\theta_{m_j}, x_1) + \max_{x_2 \in [0,\beta]} b(x_2) - c(\theta_{w_j}, x_2) = 2 \max_{y \in [0,\beta]} b(y) - c(\theta_{m_j}, y). \quad \text{(A.1)}$$

(This max exists by standard arguments.) Since $\theta_{m_j} = \theta_{w_j} \in \Theta$, Assumption 7 implies that there is an $(x'_1, x'_2) \in X$ such that $b(x'_2) - c(\theta_{m_j}, x'_1) > 0$ and $b(x'_1) - c(\theta_{w_j}, x'_2) > 0$. Hence, the first line of equation (A.1) is strictly positive, which implies that $u_m(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger) > 0$ by equation (5.2). Since the argument is analogous for a woman, we have individual rationality.

We argue that $(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger)$ cannot be blocked by contradiction. Suppose that a man m and a woman w block $(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger)$. Then, there is an $\boldsymbol{x}' = (x'_1, x'_2) \in X$ such that $u_m(w, \boldsymbol{x}') > u_m(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger)$ and $u_w(m, \boldsymbol{x}') > u_w(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger)$. Thus, $u_m(w, \boldsymbol{x}') + u_w(m, \boldsymbol{x}') > u_m(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger) + u_w(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger)$. Since (i) $u_m(w, \boldsymbol{x}') + u_w(m, \boldsymbol{x}') = b(x'_2) - c(\theta_m, x'_1) + b(x'_1) - c(\theta_w, x'_2)$ and since (ii) equation (5.2) gives $u_m(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger) + u_w(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger) = \max_{x_1 \in [0,\beta]} b(x_1) - c(\theta_m, x_1) + \max_{x_2 \in [0,\beta]} b(x_2) - c(\theta_w, x_2) = \max_{(x_1,x_2) \in X} b(x_2) - c(\theta_m, x_1) + b(x_1) - c(\theta_w, x_2)$, we have

$$b(x'_2) - c(\theta_m, x'_1) + b(x'_1) - c(\theta_w, x'_2) > \max_{(x_1,x_2) \in X} b(x_2) - c(\theta_m, x_1) + b(x_1) - c(\theta_w, x_2),$$

which is a contradiction of optimality since (x'_1, x'_2) is in X. It follows that $(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger)$ is stable.

Fourth, since Assumption 7 implies Assumption 4, Corollary 1 gives that $(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger)$ is interior and Pareto optimal.

Fifth, we establish that $(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger)$ is welfare maximizing. By equation (5.2), $T(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger) = \sum_{i \in \mathcal{N}} \max_{z_i \in [0,\beta]} b(z_i) - c(\theta_i, z_i)$. Let $(\phi', \bar{\boldsymbol{x}}')$ be a feasible allocation, let $S = \{i \in \mathcal{N} | \phi'(i) \neq i\}$ be the set of partnered players in $(\phi', \bar{\boldsymbol{x}}')$, and let $\{z'_i\}_{i \in \mathcal{N}}$ gives players' efforts in $(\phi', \bar{\boldsymbol{x}}')$. We have $T(\phi', \bar{\boldsymbol{x}}') = \sum_{i \in S} b(z'_{\phi'(i)}) - c(\theta_i, z'_i) = \sum_{i \in S} b(z'_i) - c(\theta_i, z'_i)$, where the first equality is due to the fact single players get zero and the second equality follows from rearranging the sum since $b(z'_i)$ appears in the term of $\phi'(i)$. Hence,

$$T(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger) - T(\phi', \bar{\boldsymbol{x}}') = \sum_{i \in S} (\max_{z_i \in [0,\beta]} \{b(z_i) - c(\theta_i, z_i)\} - (b(z'_i) - c(\theta_i, z'_i))) + \sum_{i \in \mathcal{N} \setminus S} \max_{y \in [0,\beta]} b(y) - c(\theta_i, y).$$

The first summand is weakly positive by optimality since $z'_i \in [0, \beta]$ for each $i \in \mathcal{N}$ by the feasibility of $(\phi', \bar{\boldsymbol{x}}')$, and the second summand is positive by Assumption 7. Thus, $T(\phi^\dagger, \bar{\boldsymbol{x}}^\dagger) - T(\phi', \bar{\boldsymbol{x}}') \geq 0$. □