WHO TO ATTACK: STABILITY PERSPECTIVES ON COORDINATION GAMES ON NETWORKS

ARKO CHATTERJEE

Extended Abstract

(This is a preliminary version of an ongoing work)

The diffusion of physical, social, cultural, digital or even conceptual substances over fixed and finite networks has been studied in great detail in a variety of fields. The influence that an entity in a social network has on its peers and vice-versa, is a fact that is well documented in the literature. Over the years, empirical evidence has been mounting in support of the perception that such effects have the potential to affect behavioural and strategic changes among the nodes of the network. On the other hand, there is a growing literature in game theory which considers coordination games with multiple Nash equilibria, being played on social networks. In a majority of these works the diffusion networks remain unchanged.

It is only in recent times that a few models pertaining to the study of diffusion under changing network topology has come up. One of these models is the node-deletion model, where the criticality of a node is measured on its ability to suffice or not suffice diffusion. The model that we consider is the exact opposite, and to the best of our knowledge, this is the first model. We consider a simple, finite, undirected network as given and define a coordination game on it. We define and characterise the pure Nash equilibria of such a network. We are interested in studying the effect on the behaviour of these original nodes of the network, when some more nodes are added to it. When does the strategy of the original nodes remain the same under the formation of the new network? When does the equilibrium of the original network remain stable in the new network? Thus, closer to the notions described by Malcolm Gladwell in the *Tipping Point*, we define a notion of the Law of the Few which dictates that for a critical number of incoming nodes, the original network no longer remains stable and tips over to a different equilibrium.

Consider a finite, simple, undirected social network $G = (V, E)$, where $V = \{1, 2, \ldots, n\}$ is the set of players and E is the set of edges in G. We define for every $i \in V$, $N(i) = \{j \in V : (i, j) \in E\}$ and $d_i = |N(i)|$. We define two games - the local game and the social game - that are played by the agents on G. The *local game* is described as follows: Each player $i \in V$ plays a $2-$person symmetric 2×2 coordination game with each of her neighbour $j \in N(i)$. The set of pure actions available to each of the two players $i \in V$ and $j \in N(i)$ is defined as $S^i = S^j = \{C, D\}$. The payoffs to (row) player i and (column) player j is given by:

	C	D
C	(a,a)	(b,c)
D	(c,b)	(d,d)

where, we assume $a > c$ and $d > b$. For the local game being played on $(i, j) \in E$, the payoff to player i is the function $u^i : S^i \times S^j \to \mathbb{R}$ and the payoff to j is $u^j : S^j \times S^i \to \mathbb{R}$. Therefore, we have, $u^i(C; C) = a$, $u^i(C; D) = b$, $u^i(D; C) = c$ and $u^i(D; D) = d$. Therefore, there are two pure strategy Nash equilibrium: (C, C) and (D, D) in the local game.

We define the *social game* on G using the local game. For every player $i \in V$, $S^i = \{C, D\}$. We define the payoff function for player $i \in V$ as $v^i : S = S^1 \times S^2 \times \cdots \times S^n = S^i \times S^{-i} \to \mathbb{R}$, such that, $v^i(s^i; s^{-i}) = \sum_{j \in N(i)} u^i(s^i; s^j)$. We say that $s^* \in S$ is the pure strategy Nash equilibrium of the social game if and only if $v^i(s^{i*}; s^{-i*}) \geq v^i(s^i; s^{-i*})\ \forall\ s^i \in S^i\ \forall\ i \in V$. Under a given configuration of the network, say $s \in S$, we define, $N^+(i) = \{j \in N(i) : s^j = C\}$, $N^-(i) = \{j \in N(i) : s^j = D\}$, $d_i^+ = |N^+(i)|$ and $d_i^- = |N^-(i)|$ for every $i \in V$, such that $d_i = d_i^+ + d_i^-$. Lastly, we assume that if for any $i \in V$ and any $s^{-i} \in S^{-i}$, $v^i(C; s^{-i}) = v^i(D; s^{-i})$, then player i always prefers the action C over D.

Proposition 1. *Suppose that $s^* \in S$ is a pure Nash equilibrium profile of the social game on the network G. Then, $\forall\ i \in V$,*

$$s^{i*} = C \textit{ if and only if } \frac{d_i^-}{d_i^+} \geq \frac{(a-c)}{(d-b)} \tag{1}$$

and,

$$s^{i*} = D \textit{ if and only if } \frac{d_i^-}{d_i^+} < \frac{(a-c)}{(d-b)} \tag{2}$$

Now, suppose that $\mathcal{I} = \{(n+1), \ldots, (n+m)\}$ is the set of m players who invade the existing social network G. We assume that every invading player, generically denoted as $\tilde{i}$, has a pre-determined action, either C or D, from which they will never deviate. We define $\mathcal{I}^+ = \{\tilde{\imath} \in \mathcal{I} : \tilde{i}\ is\ of\ type\ C\}$ and $\mathcal{I}^- = \{\tilde{\imath} \in \mathcal{I} : \tilde{i}\ is\ of\ type\ D\}$, $m^+ = |\mathcal{I}^+|$ and $m^- = |\mathcal{I}^-|$.

Thus, we have a new network $\tilde{G} = (\tilde{V} = V \cup \mathcal{I}, \tilde{E} = E \cup E_m)$, where $E_m \subseteq V \times \mathcal{I}$, is the set of new edges formed due to the invasion. We note that new edges are formed only in between an original player and an invading player. For any original player $i \in V$, we define $\tilde{N}(i) = \{\tilde{i} \in \mathcal{I} : (i, \tilde{i}) \in E_m\}$, $m_i = |\tilde{N}(i)|$, $\tilde{N}^+(i) = \{\tilde{i} \in \tilde{N}(i) : \tilde{i} \in \mathcal{I}^+\}$, $\tilde{N}^-(i) = \{\tilde{i} \in \tilde{N}(i) : \tilde{i} \in \mathcal{I}^-\}$, $m_i^+ = |\tilde{N}^+(i)|$ and $m_i^- = |\tilde{N}^-(i)|$.

For each particular definition of invasion, we will end up with a collection of new networks $\tilde{\mathcal{G}} = \{\tilde{G} : \tilde{V} = V \cup \mathcal{I}\ and\ \tilde{E} = E \cup E_m\}$. Once a new network $\tilde{G} \in \tilde{\mathcal{G}}$ is formed, the degree of an original player $i \in V$ is $(d_i + m_i)$. The payoff to a player $i \in \tilde{V}$ is defined by the payoff function $\tilde{v}^i : \tilde{S} = S \times S^{(n+1)} \times \cdots \times S^{(n+m)} \to \mathbb{R}$, where, $\tilde{S}^i = \{C, D\}$, $s^{\tilde{i}} \in S^{\tilde{i}}$ is the type of player $\tilde{i} \in \mathcal{I}$ and, $\tilde{v}^i(\tilde{s}^i; \tilde{s}^{-i}) = \sum_{j \in N(i) \cup \tilde{N}(i)} u^i(\tilde{s}^i; \tilde{s}^j)$.

We discuss three different manners of invasion - the completely deterministic manner, the partly stochastic manner and the completely stochastic manner. In each of the three cases, every invading player forms exactly one new link. In the completely deterministic case, we assume that every $i \in V$ knows m, m_i as well as m_i^+ and m_i^-. In the partly stochastic case, we assume that every $i \in V$ knows m and m_i, but do not know the values of m_i^+ and m_i^-. Naturally there are different $\tilde{G} \in \tilde{\mathcal{G}}$ possible for such specifications. In the completely stochastic case, we assume that every $i \in V$ only knows the total number of invading players m and nothing else. Therefore, in this case, any new network $\tilde{G} \in \tilde{\mathcal{G}}$ is possible. In the latter two cases, we define the expected payoff to every original player $i \in V$ as $\mathbb{E}_{\tilde{G}}[\tilde{v}^i(\tilde{s}^i; \tilde{s}^{-i})]$ and $\mathbb{E}_{\tilde{\mathcal{G}}}[\tilde{v}^i(\tilde{s}^i; \tilde{s}^{-i})]$, respectively.

For the completely deterministic case, we have the following definitions and results:

Definition 1. *Under a Nash configuration $s^* \in S$ of the network G, a player $i \in V$ with $s^{i*} = C$ is said to be (C, m_i^{+*}, m_i^{-*}) stable with respect to $\tilde{G}$, if and only if, for all $m_i^- = 0, 1, 2, \ldots, m_i^{-*}$ and for all $m_i^+ = m_i^{+*}, (m_i^{+*} + 1), \ldots, m$, $\tilde{s}^i = s^{i*}$.*

A player $j \in V$ with $s^{j*} = D$ is said to be (D, m_j^{+*}, m_j^{-*}) stable with respect to $\tilde{G}$, if and only if, for all $m_j^- = m_j^{-*}, (m_j^{-*} + 1), (m_j^{-*} + 2), \ldots, m$ and for all $m_j^+ = 0, 1, \ldots, m_j^{+*}$, $\tilde{s}^j = s^{j*}$.

Lastly, if every player $i \in V$ is stable according to the above definitions, then the Nash configuration $s^* \in S$ is defined to be stable with respect to $\tilde{G}$.

Proposition 2. *Under a Nash configuration $s^* \in S$ of the original network G, a player $i \in V$ with $s^{i*} = C$ is (C, m_i^{+*}, m_i^{-*}) stable if and only if,*

$$\frac{(d_i^- + m_i^{-*})}{(d_i^+ + m_i^{+*})} \leq \frac{(a-c)}{(d-b)} \tag{3}$$

and, a player $j \in V$ with $s^{j*} = D$ is (D, m_j^{+*}, m_j^{-*}) stable if and only if,

$$\frac{(d_j^- + m_j^{-*})}{(d_j^+ + m_j^{+*})} > \frac{(a-c)}{(d-b)} \tag{4}$$

Corollary 1. *A Nash configuration $s^* \in S$ is (m^{+*}, m^{-*}) stable with respect to a new network $\tilde{G} \in \tilde{\mathcal{G}}$, if and only if,*

$$\max_{i:s^{i*}=C} \Big[\frac{(d_i^- + m_i^{-*})}{(d_i^+ + m_i^{+*})}\Big] \leq \frac{(a-c)}{(d-b)} \tag{5}$$

and,

$$\min_{i:s^{i*}=D} \Big[\frac{(d_i^- + m_i^{-*})}{(d_i^+ + m_i^{+*})}\Big] > \frac{(a-c)}{(d-b)} \tag{6}$$

For the partly stochastic case, we have the following definitions and results:

Definition 2. *Under a Nash configuration $s^* \in S$ of the network G, a player $i \in V$ with $s^{i*} = C$ is said to be (C, m^{+*}, m^{-*}) stable with respect to all the possible $\tilde{G}$, if and only if, for all $m^- = 0, 1, 2, \ldots, m^{-*}$ and for all $m^+ = m^{+*}, (m^{+*}+1), \ldots, m$, $\mathbb{E}_{\tilde{G}}[\tilde{v}^i(C; \tilde{s}^{-i*})] \geq \mathbb{E}_{\tilde{G}}[\tilde{v}^i(D; \tilde{s}^{-i*})]$.*

A player $j \in V$ with $s^{j*} = D$ is said to be (D, m^{+*}, m^{-*}) stable with respect to all the possible $\tilde{G}$, if and only if, for all $m^- = m^{-*}, (m^{-*}+1), (m^{-*}+2), \ldots, m$ and for all $m^+ = 0, 1, \ldots, m^{+*}$, $\mathbb{E}_{\tilde{G}}[\tilde{v}^j(C; \tilde{s}^{-j*})] < \mathbb{E}_{\tilde{G}}[\tilde{v}^j(D; \tilde{s}^{-j*})]$.

Lastly, if every player $i \in V$ is stable according to the above definitions, then the Nash configuration $s^* \in S$ is defined to be stable with respect to all the possible $\tilde{G}$.

Proposition 3. *Under a Nash configuration $s^* \in S$ of the original network G, a player $i \in V$ with $s^{i*} = C$ is (C, m^{+*}, m^{-*}) stable if and only if,*

$$\frac{(md_i^- + m^{-*}m_i)}{(md_i^+ + m^{+*}m_i)} \leq \frac{(a-c)}{(d-b)} \tag{7}$$

and, a player $j \in V$ with $s^{j*} = D$ is $(-1, m^{+*}, m^{-*})$ stable if and only if,

$$\frac{(md_j^- + m^{-*}m_j)}{(md_j^+ + m^{+*}m_j)} > \frac{(a-c)}{(d-b)} \tag{8}$$

Corollary 2. *A Nash configuration $s^* \in S$ is (m^{+*}, m^{-*}) stable with respect to the different possible new networks $\tilde{G} \in \tilde{\mathcal{G}}$, for which only m_i is known for all $i \in V$ apart from the knowledge of the original network G, if and only if,*

$$\max_{i:s^{i*}=C} \Big[\frac{(md_i^- + m_i m^{-*})}{(md_i^+ + m_i m^{+*})}\Big] \leq \frac{(a-c)}{(d-b)} \tag{9}$$

and,

$$\min_{i:s^{i*}=D} \Big[\frac{(md_i^- + m_i m^{-*})}{(md_i^+ + m_i m^{+*})}\Big] > \frac{(a-c)}{(d-b)} \tag{10}$$

For the completely stochastic case, the definitions of stability is exactly the same as Definition 2, except for the fact that the notion of expected payoff is now changed from $\mathbb{E}_{\tilde{G}}[\tilde{v}^i(\tilde{s}^i;\tilde{s}^{-i})]$ to $\mathbb{E}_{\tilde{\mathcal{G}}}[\tilde{v}^i(\tilde{s}^i;\tilde{s}^{-i})]$. We have the following results:

Proposition 4. *Under a Nash configuration $s^* \in S$ of the original network G, a player $i \in V$ with $s^{i*} = C$ is (C, m^{+*}, m^{-*}) stable if and only if,*

$$\frac{(nd_i^- + m^{-*})}{(nd_i^+ + m^{+*})} \leq \frac{(a-c)}{(d-b)} \tag{11}$$

and, a player $j \in V$ with $s^{j*} = D$ is (D, m^{+*}, m^{-*}) stable if and only if,

$$\frac{(nd_j^- + m^{-*})}{(nd_j^+ + m^{+*})} > \frac{(a-c)}{(d-b)} \tag{12}$$

Corollary 3. *A Nash configuration $s^* \in S$ is (m^{+*}, m^{-*}) stable with respect to $\tilde{\mathcal{G}}$, if and only if,*

$$\max_{i:s^{i*}=C} \left[\frac{(nd_i^- + m^{-*})}{(nd_i^+ + m^{+*})}\right] \leq \frac{(a-c)}{(d-b)} \tag{13}$$

and,

$$\min_{i:s^{i*}=D} \left[\frac{(nd_i^- + m^{-*})}{(nd_i^+ + m^{+*})}\right] \geq \frac{(a-c)}{(d-b)} \tag{14}$$

The Corollaries 1, 2, and 3 can be understood through the lens of the Law of the Few. This immediately leads us on to the crucial questions of how can the above results be utilised to study diffusion of a particular strategy through a network, using the concept of invading nodes. Furthermore, we are also investigating the effect that various centrality measures might have in defining the appropriate notions of the Law of the Few.