Learning and Convergence to Nash in Network Games with Continuous Action Set

Preliminary Draft

Sebastian Bervoets* Mario Bravo† and Mathieu Faure‡

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Abstract

In this paper we study a simple learning process for games that have continuous action sets and that are played on a network, allowing for heterogeneous patterns of interactions. Our learning process assumes that agents are unsophisticated and ignore everything of the game, as well as of the structure of the network. They only observe their own payoff after actions have been played and choose their next action according to how their payoff has varied. We also assume that players update their actions simultaneously. We show that in a very large class of games, convergence to Nash happens much more easily than in discrete games, thus providing some foundations to the use of Nash equilibrium. In particular, we show that in generalized ordinal potential games, convergence to Nash happens with probability one. Moreover, we show that, unless the network is bipartite, our process never converges to a unstable equilibrium. We also examine games with strategic complements and show that the process converges to a stable equilibrium with probability one, for non bipartite networks.

1 Introduction

In this paper we study the issue of learning in games where agents have a continuous set of strategies to choose from and know as little as possible on the game they are

*Aix-Marseille University (Aix-Marseille School of Economics), CNRS and EHESS

†Universidad de Santiago de Chile, Departamento de Matemática y Ciencia de la Computación

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playing. We investigate whether agents can converge to Nash when using very simple behaviors. We do this in the general context of games played on a network (also called graphical games), thereby allowing for heterogeneous interactions between agents, but also encompassing standard games where agents interact with every other agent, when the network is complete.

When the game has Nash equilibria, agents need a lot of information to compute them: their own payoff function, the set of opponents they are facing, the payoff function of their opponents. In case of a game played on a network, agents must additionally know precisely what is the structure of the network. With all this information, agents can compute everyone's best-response function and potentially determine the set of equilibria. Then, there is the issue of coordination on a specific equilibrium. In real-life situations, however, it is very unlikely that agents are so sophisticated.

The ubiquity of the concept of Nash equilibrium in game theory, and in economics in general, has thus led researchers to study the convergence of learning processes, when agents are not fully informed on the game they are playing. This literature has provided a lot of insightful results about the convergence to Nash, be it the Nash strategy profile or the Nash payoff, in the context of discrete games. Several learning procedures (such as fictitious play or reinforcement learning[1]) have been analyzed in standard games (such as global sum games for instance), often with only two players. Even though the literature is growing and more general results are expected, the main message coming out of this literature is that in discrete games, convergence to Nash is very difficult to obtain. Nothing gets simpler if a network structure is added to the game.

In this paper, we suggest that convergence to Nash is easier to obtain in games with continuous action sets, for very large classes of games, with arbitrary numbers of players, even when we assume minimal knowledge of the game and when we allow for heterogeneous interaction patterns, modeled by a network. The only assumption we make on the games we analyze is that the payoff functions are strictly concave in own action[2]. Because in many economic contexts, agents face a continuous action set (prices, effort levels etc.), and the strict concavity is satisfied, we believe our results provide some strong foundations to the use of Nash equilibrium concept in economics.

Let us describe the learning process we consider. The continuous game is repeated, and at each step, all players simultaneously choose a direction at random (higher or lower action), and move a small amount away from their current strategy, in that direction. We call this the exploration stage. Players observe the new payoff associated to this new action and compare it with the payoff they obtained in the previous period.

[1]In fictitious play, players are assumed to know their payoff function, and although they ignore their opponents' payoff function, they still observe what actions they have played. With reinforcement learning, players ignore everything about the game they are playing, they just observe the payoff they receive after taking an action, and map this payoff into probabilities of taking actions in the next step. These learning procedures are difficult to adapt to continuous action spaces.

[2]This assumption implies that, once the opponents' actions are fixed, the individuals' payoff functions have a unique maximum.

If the payoff has increased, players move away in the same direction; if the payoff has decreased, players move in the opposite direction. The amplitude of an individual's change in action is assumed to be proportional to the change of that individual's payoff during the exploration stage. We call this the updating stage. After the updating stage, a new exploration stage begins. We study the convergence of the updating stages.

This learning procedure requires almost no sophistication from the agents, as they only observe their own payoffs of the current and the previous period, and choose a new action, regardless of what mechanism has produced these payoffs (in particular, they ignore their payoff function but also ignore anything that relates to opponents - if there is a network, they do not even know who are their opponents).

The process has a stochastic part, because during the exploration stage the direction of change is chosen at random. Thus, we resort to stochastic approximation theory to analyze the convergence of the process. This theory tells us that, under some standard condition on the noise introduced by the random component, the behavior of the stochastic process is driven by an underlying deterministic dynamic system. We thus relate the set of points to which the random process may converge (the *limit set*) to the set of points to which the deterministic dynamic system converges (the ω-*limit set*).

Our first set of results is very general. It holds for any game that is strictly concave, and is independent of the structure of the network it is played on: first, the set of Nash equilibria of the game is included in the set of stationary points of the dynamics; second, if the learning process converges to a zero of the dynamic system, then it has to be a Nash equilibrium; third, if some Nash equilibria are asymptotically stable[3], then the process will converge to one of these with positive probability.

Taken together, these results tell us a lot and not much at the same time. On the one hand side, at this level of generality, it is surprising that such statements can be made. Agents playing a game of which they ignore everything, and following a simple updating process, can end up playing a stable Nash equilibrium. And this is true for a very general class of games. However, the process could also converge to sets that are not Nash equilibria; and in case it does converge to a Nash equilibria, it could converge to an unstable one.

Our second general result states that, whether the process may or may not converge to an unstable Nash equilibrium, depends on the structure of the network. We show that when the game exhibits symmetric externalities[4], then the process can converge to an unstable equilibrium only if the network is a bipartite network. This is because for bipartite networks it could be the case that the stochastic noise, which is correlated between agents through externalities, could cancel out in the vicinity of an unstable point. As a consequence the noise might not be able to drive the process in the unstable

[3]By asymptotically stable Nash equilibria, we mean that it captures any trajectory that passes close to it. We will be more specific in the text.

[4]By symmetric externalities, we mean that if individual i exerts a positive (resp. negative) externality on individual j, then j exerts a positive (resp. negative) externality on i - though not necessarily of the same intensity.

direction. On the contrary, when the network is not bipartite, there is at least a triangle of agents for which externalities do not balance, which implies that the noise cannot cancel in the neighborhood of an unstable point. Thus the process will always take the unstable direction at some moment, and leave the unstable equilibrium.

In the second part of the paper, we provide more precise convergence results by focusing on two broad classes of games: the class of ordinal potential games and the games with strategic complements[5].

The class of ordinal potential games is of interest to the network literature because, as we prove, it contains all the local aggregate games, where individual payoffs depend on own action as well as on the sum of neighbors' action. These games include the standard network games such as the game of linear complementarities (Ballester et al. (2006)), the public good game (Bramoullé and Kranton (2007) and Bramoullé et al. (2014)), the public good with private provision game (Allouch (2015)) and oligopoly games, including Cournot competition (Bimpikis et al. (2014), König et al, 2014). In all of these games, Nash equilibria have been studied and depend on the structure of the network, inducing another dimension of complexity for unsophisticated agents[6].

For this class of games, we get a drastic improvement: we show that our learning process converges to the set of Nash equilibria, with probability one. Any other limit set of the stochastic process, as well as any other ω-limit set of the dynamic system, can thus be discarded. This is a very strong result, in particular because it is independent of the structure of the network. Therefore, the positive results obtained justify even more the use of Nash equilibrium concepts in the network literature. We also show that it is possible to distinguish between equilibria that are stable for our process and those that are not, by making use of the properties of the ordinal potential of the game. However, we cannot exclude that the process will converge to a set containing some unstable equilibria, even when the graph is not bipartite.

Finally, in games with strategic complements, we show that the process cannot converge to an unstable Nash equilibrium, regardless of the structure of the network. Moreover our process converges to a stable Nash equilibrium with probability one, when the network is not bipartite. This again is a very strong result, since the learning process not only guarantees convergence to Nash, it also serves as a selection device between stable and unstable equilibria.

Related Literature

As mentioned earlier, in the learning literature for finite action games, Fictitious Play (introduced in Brown (1951)) is possibly the most studied adaptive process in game the-

[5]Notice that some games with strategic complements fall into the class of ordinal potential games and obviously, some ordinal potential games are not with strategic complements.

[6]For instance, in the game introduced by Ballester, Calvó-Armengol and Zenou, the Nash equilibrium is related to the Bonacich centrality of agents, which are very difficult to compute because they are infinite sums of discounted paths of the network.

ory. The procedure in itself is very simple: at each step players compute a best-response to the opponent's empirical average moves, and play this action. This process is shown to converge in various classes of games: see Robinson (1951) for 2-player zero-sum games, Miyazawa (1961) for 2×2 games, Monderer and Shapley (1996a) for potential games, and Berger (2005) for $2 \times N$ games. Stochastic approximation theory has been successfully used to generalize some of these results: see Hofbauer and Sorin (2006) for more general results on zero-sum games and Benaïm et al. (2005) for nonlinear payoff functions on compact action sets. Most of these convergence results also hold in some sense for *stochastic fictitious play*, introduced by Fudenberg and Kreps (1993), (see also Fudenberg and Levine (1998)), where agents use a fictitious play strategy in a game where payoff functions are perturbed by random variables, in the spirit of Harsanyi Harsanyi (1973): convergence for this procedure holds in 2×2 games (see Benaïm and Hirsch (1999)), zero-sum and potential games (see Hofbauer and Sandholm (2002)) and supermodular games (see Benaïm and Faure (2012))

Still considering finite games, fictitious play does not apply when agents do not observe their opponents' actions and/or do not necessarily know their own payoff function. In these settings, *Reinforcement Learning Procedures* are the most popular processes (see Börgers and Sarin (1997) or Erev and Roth (1998) for pioneer work). Again the idea behind these procedures is quite simple, but requires two steps at each stage: some state variable that aggregates all the available information up to date is updated and some action is chosen taking into account this variable and possibly the previous moves. Reinforcement learning processes have been successfully used in simple games: 2×2 games in Posch (1997), 2-player games with positive payoffs in Börgers and Sarin (1997), Beggs (2005) or Hopkins and Posch (2005). On the same topic, see also Leslie and Collins (2005) and Cominetti et al. (2010).

However, the standard approaches just described can hardly be adapted to continuous games. It is worthwhile noting that the stochastic fictitious play has been adapted by Perkins and Leslie (2014) to continuous actions games, but besides the complexity of this adapted process, the fictitious play algorithm assumes that agents have some knowledge of the game. We do not want to assume this here as we consider unsophisticated agents.

In the context of continuous action space, only two types of models have been explored. First, there are models of best-reply dynamics, with discrete or continuous time versions, where at each period agents play their deterministic best-response to what the other agents played in the previous period. However, the information needed to compute best-responses (own payoff function and actions taken by the opponents) might be unrealistic.

Second, a scarcer literature assumes stochastic adjustment processes where agents know very little about the game. This is the case of better-reply dynamics for instance (see Dindos and Mezzetti, 2006), where at each step of the dynamics, an agent plays a strategy chosen at random, observes the hypothetical payoff that this action yields, and then decides whether to stick to this new strategy or to come back to the previous one, depending on whether the payoff has increased or not. The process then goes on. Although little knowledge of the game is necessary to implement better-reply dynamics, a crucial assumption is that agents revise their strategy sequentially. At each step, one and only one player is randomly chosen and can experiment some new strategy, while all other players keep their strategy unchanged. The driving force for convergence is that with some positive probability, every player will be randomly drawn as many times as necessary to approximate a best-response. The proof techniques then consist of showing that best-responses are well-enough approximated to guarantee convergence. However, one can easily construct a simple game where simultaneity drives to non-convergence.

Our process differs from previous ones in several aspects. First, agents only need a minimal amount of information. They observe their own current payoff and the payoff of the previous period and make decisions, without knowing anything about their opponents, nor do they know what process generates the payoffs. This discards reasoning based on best-responses. Second, players all move simultaneously. This makes it more realistic since in real-life, agents do not play one after the other. It also discards reasoning based on approximated best-responses. Third, we do not restrict our attention to games played by two players, instead we consider games played by any finite number of agents. Fourth, we consider games played on a network, thereby allowing for heterogeneous interactions. Standard games then come as a special case where the network is complete (and all agents interact with all other agents), but we do not restrict our attention to these cases. Indeed, in real-life situations, individuals are embedded in social networks and interact only with their neighbors.

The rest of the paper is organized as follows: we introduce the model in section 2 and prove that our learning process is a discrete time stochastic process. In section 3, we give general results about convergence, before turning to the case of ordinal potential games in section 4 and games with strategic complements in section 5. All our proofs are in the appendix.

2 The model

The Game

We consider a repeated game $\mathcal{G} = (\mathcal{N}, X, u)$, where $\mathcal{N} = \{1, \ldots, N\}$ is the set of players, $X_i \subset \mathbb{R}$ is a continuous action space and $X = \times_{i=1,\ldots,N} X_i$. An action $x_i \in X_i$ can be thought of as an effort level chosen by individuals, a price set by a firm, a monetary contribution to a public good... Because in all these examples actions take positive values, we will generally consider $X_i = [0, +\infty[$. Finally, $u = (u_i)_{i=1,\ldots,N}$ is the vector of payoff functions, where u_i will enjoy various properties in different sections of the paper. However, we will always make the following standing assumption:

Hypothesis 1 *u_i is assumed to be twice-differentiable on $\mathbb{R}_+^N$ and strictly concave in x_i: $\frac{\partial^2 u_i}{\partial x_i^2}(x) < 0$.*

Notice that hypothesis 1 implies that Du_i is bounded everywhere. For simplicity and without loss of generality, in what follows we will assume that $|u_i(x) - u_i(x')| < \|x - x'\|_\infty$[7].

While for most of our results, we need no further assumption, in some cases we will restrict our attention to games that exhibit symmetric externalities.

Definition 1 (Symmetric externalities) *We say that the game has* symmetric externalities *if $\forall i \neq j$, $\forall x$, we have*

$$sgn\left(\frac{\partial u_i}{\partial x_j}(x)\right) = sgn\left(\frac{\partial u_j}{\partial x_i}(x)\right).^{8}$$

As we will see later, most of the games in the network literature fall into this class. Note that a game with symmetric externalities does not impose that externalities are of equal intensity, they need just be of the same sign. Note also that symmetric externalities allow for interactions where i exerts a positive externality on individual j and a negative externality on individual k. Note finally that games with symmetric externalities do not imply that $sgn\left(\frac{\partial u_i}{\partial x_j}(x)\right) = sgn\left(\frac{\partial u_i}{\partial x_j}(x')\right)$. Thus signs may change when x changes.

The Network

A game describes a pattern of interactions between pairs of individuals. We thus define an interaction graph as the graph induced by the *interaction matrix* $\mathbf{G}(x)$ defined as: $g_{ii}(x) = 0$ and, for $i \neq j$, $g_{ij}(x) = 0$ whenever there exists a neighborhood U of x such that $\frac{\partial u_j}{\partial x_i} = 0$ on U. Otherwise, $g_{ij}(x) = 1$.

Note that the interaction matrix is defined in the most general way, as it depends on the action profile x. Thus it can change as actions change. However, as we will see, in the economic models we have in mind, the interaction graph is independent of the action profile. In that case we will say that our game is a *network game*:

Definition 2 *A game $\mathcal{G}$ is a network game if, for any $x, x' \in X$ and any $i \neq j$, we have $g_{ij}(x) = g_{ij}(x')$.*

Hence, in a network game, individuals that are not linked are those that never exert any externality on each other. In a network game, we can thus drop the x from the

[7]This is just meant to guarantee that our learning process is well behaved, although it can easily be adapted when it does not hold

[8]Where $sgn(a) = 0$ if $a = 0$

notation and write $g_{ij} = 0$ or $g_{ij} = 1$, as is standard in the network literature. Note that, as usual, externalities can flow through the network, and even though $g_{ij} = 0$, individual j's action will affect i's payoff at equilibrium when the network is connected, through the path of individuals that link i to j.

The Learning Process

Agents aim at maximizing their payoff by choosing an action, but they ignore the payoff functions $(u_i)_i$, as well as the set of players. They observe the realization of their payoffs after every period and choose a new action using an updating rule that we describe in words before presenting it formally. We describe what agent i does, but keep in mind that every agent simultaneously uses the same updating rule.

• At the beginning of a round, agent i is playing action x_i and is enjoying the associated payoff $u_i(x_i, x_{-i})$. Every round of the game is subdivided into two periods.

• In the first period, player i tests a new action x'_i that is very close to his current action x_i. By very close we mean that the magnitude of the deviation $|x'_i - x_i|$ is decreasing and tends to zero as rounds of the game unfold. Whether this test action is higher or lower than his current action is drawn at random, each with probability one half. Agent i then discovers the new payoff $u_i(x'_i, x'_{-i})$, and this ends the first period.

• In the second period, agent i chooses an action x''_i depending on the outcome of the first period. If $u_i(x'_i, x'_{-i})$ is higher than $u_i(x_i, x_{-i})$, then agent i enjoyed the change and moves in the same direction as in the first period. On the contrary, if $u_i(x'_i, x'_{-i})$ is lower than $u_i(x_i, x_{-i})$, then agent i disliked the change and moves in the opposite direction. The magnitude by which agent i changes his action depends on the magnitude of the payoff variation: if increasing action from x_i to x'_i has resulted in a small increase (resp. decrease) of his payoff, agent i will choose an action x''_i that is higher (resp. lower) than, but not far away from x_i. If on the contrary, it has resulted in a large increase (resp. decrease) of his payoff, agent i will choose an action x''_i that is far higher (resp. lower) than x_i.

• At the end of the second period, agent i gets $u_i(x''_i, x''_{-i})$ and the round ends. A new round starts with x'' as the new action profile.

Formally, let $(\epsilon^i_n)_n$ be a sequence of i.i.d random variables such that $\mathbb{P}(\epsilon^i_n = 1) = \mathbb{P}(\epsilon^i_n = -1) = 1/2$. Initially agent i chooses an action e^i_0. Then at each stage $n \geq 0$, player i selects his actions at stages $2n+1$ and $2n+2$ the following way:

(*i*) ϵ^i_n is drawn and player i chooses $e^i_{2n+1} := e^i_{2n} + \frac{1}{n+1}\epsilon^i_n$; (*exploration stage*)

(*ii*) agent i plays $e^i_{2n+2} := e^i_{2n}(1 + \epsilon^i_n \Delta u^i_{n+1})$, where (*updating stage*)

$$\Delta u^i_{n+1} := u_i(e^i_{2n+1}, e^{-i}_{2n+1}) - u_i(e^i_{2n}, e^{-i}_{2n})$$

Let $x_n = e_{2n}$ and $\mathcal{F}_n$ be the history generated by $\{e_1, ..., e_{2n+1}\}$. Studying the asymptotic behavior of the random sequence $(e_n)_n$ amounts to studying x_n. Hence the main focus of this paper is on the convergence of the random process $(x_n)_n$.

Remark 1 *Notice that in the updating stage, we multiply Δu^i_{n+1} by e^i_{2n}. This is because at each step of the process, we need to make sure that actions stay within the bounds (positive, that is). It is therefore important that the change in agents' actions dampen down as they approach* 0*, in order to stay positive. One natural way of doing that is to add a factor that takes smaller and smaller values as actions go closer to* 0*, as we did. At this stage, it is not obvious that actions remain positive on exploration stages. We show in the appendix (lemma 2) that indeed, this is the case if initial conditions are chosen widely, so our process is well-defined.*

The next lemma is fundamental for the rest of our analysis.

Lemma 1 *The iterative process can be written as*

$$x_{n+1} = x_n + \frac{1}{n+1}\left(G(x_n) + U_{n+1} + \xi_{n+1}\right) \tag{1}$$

where

(i) $G(x) = (G_i(x))_i$ *with* $G_i(x) = x_i \frac{\partial u_i}{\partial x_i}(x_i, x_{-i})$

(ii) U_{n+1} *is a bounded martingale difference (i.e* $\mathbb{E}\left(U_{n+1} \mid \mathcal{F}_n\right) = 0$*)*

(iii) $\xi_n = \mathcal{O}(1/n)$.

The iterative process (1) is a discrete time stochastic process with step size satisfying $\sum_n \frac{1}{n} = \infty$ and $\lim_{n\to\infty} \frac{1}{n} = 0$.[9] If there were no stochastic term, the process (1) would write

$$x_{n+1} = x_n + \frac{1}{n+1} G(x_n)$$

which corresponds to the well-known Euler method, a numerical procedure for approximating the solutions of the deterministic ordinary differential equation (ODE)

$$\dot{x} = G(x) \tag{2}$$

[9]These two observations are crucial. It is important that the sum diverges, to guarantee that the process does not get "stuck" anywhere. If the sum was converging, then after some time the agents would stay where they are, but we would be unable to distinguish whether they stay there because they like it, or because they cannot move. Further, it is important that the terms go to zero, so that the process can "settle" when agents want to. Having a term that does not tend to zero would oblige agents to move a lot all the time, even when they would like to stay where they are. Convergence would then never be observed.

Of course, the (stochastic) process (1) differs from the (deterministic) process (2), because (1) has a stochastic noise. However, as we have just proved, this noise is a martingale difference and this guarantees that, on average, the noise will be zero. This allows us to use stochastic approximation theory to relate the behavior of $(x_n)_n$ to the flow of the ODE (2). This method, which is often referred to as *the ODE method*, was initially introduced in Ljung (1977) and has been extended by many authors (see for instance Kushner and Clark (1978), Benveniste et al. (1990), Duflo (1996) or Kushner and Yin (2003)) for simple dynamics (typically linear or gradient-like systems). In a series of papers, Benaïm and coauthors introduced a more general approach that allows to describe the asymptotic behavior of the random process $(x_n)_n$, beyond gradients and linear dynamics. Most of the results in this direction can be found in Benaïm (1999).

Remark 2 *As we will detail these relations between the stochastic process and the deterministic counterpart, we will also stress that the dynamical system (2) shares some features with the continuous-time best-response dynamics (BRD), introduced by Gilboa and Matsui (1991) and defined in the standard way:*

$$\dot{x} = -x + BR(x) \tag{3}$$

(BRD) has been extensively studied (see Matsui (1992), Hofbauer (1995) or Hofbauer and Sorin (2006)), and although the underlying dynamics in our learning process has little in common conceptually as solution curves may be very different, it does share some qualitative properties. As a by-product of our analysis, our process therefore provides some insights on the best response dynamics.

Stability concepts

In the rest of the paper, we will focus on stationary points of the dynamical system (2), as we will show that Nash equilibria are included in this set. We wish to distinguish between stationary points that are stable vis à vis our dynamics, and those that are unstable. Because no specific structure is imposed on the games we consider, the reader should keep in mind that stationary points of the dynamical system, and in particular Nash equilibria, can either be *isolated* or lie in a *continuum*.

In order to illustrate many of the concepts that we refer to, as well as some of our results, we present a very simple and standard economic example, i.e. Cournot competition on a networked market.

Example 1 *Consider a set of firms localized on a geographical network G, all producing the same good and competing against their neighbors. Let $x_i \in \mathbb{R}^+$ denote the quantity produced by firm i at a constant marginal cost $c > 0$, that they will sell at price p_i, and let α and γ be some positive real numbers. Denote $r \equiv \frac{\alpha - c}{\gamma}$ and assume that the*

(a) Isolated Nash (b) Continuum of Nash, $\alpha \in [0, r]$

Figure 1: Different Types of Nash equilibria (black dots correspond to effort r, while white dots correspond to null effort

parameters are such that $r > 0$. *Finally, let* $\bar{x_i}$ *be the aggregate quantity produced by i's neighbors:* $\bar{x}_i = \sum_{\{j;g_{ij}=1\}} x_j$. *The market price is given by:*

$$p_i(x) = \alpha - \frac{\gamma}{2}x_i - \gamma\bar{x}_i$$

and firm i's profits are given by:

$$u_i(x) = p_i(x)x_i - cx_i$$

This game satisfies hypothesis 1, is a network game and exhibits symmetric externalities. The first order conditions imply that, at equilibrium:

$$x_i^* = Max\{r - \bar{x_i^*}, 0\}$$

In this simple game, many equilibria structures can appear. Consider for instance the network with 4 agents and just one missing link between agents 1 *and* 3, *featured in Figure 1. Then* $(r, 0, r, 0)$ *is an equilibrium of the game, and it is isolated, as shown in panel 1a. However there is also a continuum of Nash equilibria given by* $(0, \alpha, 0, r - \alpha)$, *for* $\alpha \in [0, r]$, *as in panel 1b.*

If we drop the link between 2 *and* 4, *as in Figure 2 then the profile* $(r, 0, r, 0)$ *is still an isolated equilibrium, but now there is also an interior equilibrium which is isolated, given by* $(\frac{r}{3}, \frac{r}{3}, \frac{r}{3}, \frac{r}{3})$.

When x is isolated, we will say that x is (un)stable if it is linearly (un)stable:

Definition 3 *Given* $x \in X$, *let* $DG(x)$ *be the Jacobian matrix of* G *evaluated at* x. *Then* x *is a linearly unstable equilibrium if*

$$\exists \lambda \in Sp(DG(x)) \;\; s.t. \quad Re(\lambda) > 0.$$

while it is linearly stable if

$$\forall \lambda \in Sp(DG(x)) \;\; s.t. \quad Re(\lambda) < 0.$$

(a) Isolated specialized Nash (b) Isolated Interior Nash

Figure 2: Different Isolated Nash equilibria

However, when x is in a continuum, the notion of linear stability has no bite, in particular because generally every Nash equilibrium x in a continuum has a Jacobian with some eigenvalue that is 0. Thus, for components of Nash equilibria, we resort to another notion of stability. We will say that a component of Nash equilibria is stable if it is an attractor of the dynamical system (see Ruelle (1981)):

Let $\varphi(x,t)$ denote the flow of $G(.)$, i.e. $\varphi(x,t)$ is the solution of (2) with initial condition x, at time t[10].

Definition 4 *Let S be a compact subset of $\mathbb{R}^N$. Then S is invariant for the flow φ if*
a) $\forall x \in S$, $\forall t \in \mathbb{R}$, $\varphi(x,t) \in S$ and
b) $\forall y \in S, \forall t \in \mathbb{R}$, there exists $x \in S$ such that $\varphi(x,t) = y$

Condition a) states that if the initial condition is within the invariant set, then the flow should stay inside the invariant set. Condition b) states that for any point y of the invariant set, we can find an initial condition inside the set that leads to y in a given time t.

Definition 5 *Let S be invariant for the flow φ. Then a set $A \subset S$ is an attractor for $\dot{x} = G(x)$ if*
(i) A is compact and invariant;
(ii) there exists an open neighborhood U of A with the following property:

$$\forall \epsilon > 0,\ \exists T > 0 \text{ such that } \forall x \in U,\ \forall t \geq T,\ d(\varphi(x,t), A) < \epsilon.$$

An attractor for a dynamical system is a set with strong properties. In particular, it uniformly attracts a neighborhood of itself. Indeed, if the dynamics starts inside of the attractor, it will stay there. Furthermore, if we perturb slightly the vector x so as to eject it outside of the attractor, it will come back to it.

Remark 3 *An isolated equilibrium is an attractor if and only if it is linearly stable.*

[10]Notice that by the regularity assumption on $u(.)$, G satisfies the Cauchy-Lipschitz condition that guarantees that for all $x \in X$, $\varphi(x,t)$ is unique.

In the remainder, when we mention that a component of equilibria is stable, we refer to the fact that it is linearly stable if it is degenerate (i.e. an isolated equilibrium), and to the fact that it is an attractor otherwise.

3 General results using stochastic approximation theory

This section is dedicated to stating convergence results for very general games. Relying on the existing literature, not much can be said about our random process $(x_n)_n$ in full generality. The main result that we can use says the following: *with probability one, either $(x_n)_n$ is unbounded or its limit set is compact, invariant and attractor-free* (see Benaïm (1996) or Benaïm (1999))[11]. This result does not inform us on the question we ask about convergence to Nash. In this section we thus provide some results that are specific to our random process. Let us define some convergence notions first.

Definition 6 (Limit set of $(x_n)_n$) . *Let ω be a realization of the random process.*

$$\mathcal{L}\left((x_n)_n(\omega)\right) := \{x \in S;\ \exists \textit{ a subsequence } x_{n_k} \textit{ such that};\ x_{n_k}(\omega) \to x\} \tag{4}$$

Note that the limit set of the learning process is a random object, because the asymptotic behavior of the sequence $(x_n)_n$ depends on the realization of the ransom sequence $(\epsilon_n)_n$. By abuse of notation we drop the ω from the notation and write the limit set as $\mathcal{L}(x_n)_n$.

Now we define the analogous concept for the trajectories of the dynamical system (2):

Definition 7 (ω-limit set) *Let $x \in X$. The omega limit set of x is*

$$\omega(x) := \{z \in X;\ lim_{k\to\infty}\varphi(x, t_k) = z \textit{ for some } t_k \to \infty\}.$$

All ω-limit sets of the ODE are compact, invariant and attractor-free (see Benaïm (1999) for instance). Thus they are of particular interest to us, because they are candidates to be limit sets of our stochastic process (1).

These sets can have complex properties, but among them are the zeroes of the dynamical system, defined as follows: $Z(G) := \{x \in X; G(x) = 0\}$. Because $G_i(x) = x_i \frac{\partial u_i}{\partial x_i}(x)$, we have:

$$x \in Z(G) \Longleftrightarrow \forall i \in N, \left(x_i = 0 \ \text{ or } \ \frac{\partial u_i}{\partial x_i}(x_i, x_{-i}) = 0\right)$$

Let NE denote the set of Nash Equilibria of the game $\mathcal{G}$. Then we have the following:

[11] At this point, we are fully aware the answer will not be very enlightening for a reader that is not familiar with this theory. However we give a formal statement in appendix, along with some examples.

Proposition 1 $x \in NE \Longrightarrow x \in Z(G)$

This result, although straightforward, is crucial to what we are doing. Indeed, it states that every component of Nash equilibria of the game, whether degenerate or not, is an ω-limit set of the dynamical system, and as such, is a candidate to be a limit set of the process. Therefore our process does not *ex ante* reject any equilibria of the game.

To see why proposition 1 holds, observe that since $X_i = [0; +\infty[$, we have

$$x \in NE \Longleftrightarrow \;\forall i \in N, \left(\frac{\partial u_i}{\partial x_i}(x_i, x_{-i}) = 0 \text{ or } \; x_i = 0, \text{ and } \; \frac{\partial u_i}{\partial x_i}(x_i, x_{-i}) \leq 0 \right)$$

As a consequence, we can see that NE is not the only set contained in $Z(G)$:

$$Z(G) = NE \cup \left\{ x : \; G(x) = 0 \text{ and } \exists i \; \text{ s.t. } x_i = 0, \; \frac{\partial u_i}{\partial x_i}(x) > 0 \right\}.$$

We call $x \in Z(G) \setminus NE$ an *other zero* (OZ) of the dynamical system: $Z(G) = NE \cup OZ$. Points that are in OZ are such that at least one agent would like to deviate.

To illustrate, take the Cournot competition model presented in example 1, on the network in Figure 1. Then the action profile $(0, \alpha, \beta, r - \alpha - \beta)$ with $\beta > 0$ and $\alpha + \beta \leq r$ is a component of OZ, in particular because agent 1 would get a higher payoff by increasing his action. Furthermore, this component of OZ is connected to a component of NE, by setting $\beta = 0$. In the square (Figure 2), the action profile $(\alpha, r - \alpha, 0, 0)$ with $\alpha \in [0, r]$ is also a component of OZ, but it is not connected to any NE. Last, in a 3 agents line, the profile $(\alpha, r - \alpha, 0)$ lies in a continuum of OZ and it is connected with a NE, when $\alpha = 0$, but this NE is isolated.

Remark 4 *Other zeroes are linearly unstable (see proof in the appendix).*

Our first result states that the process (1) will never converge to an element of OZ.

Theorem 1 *We have that*

$$\mathbb{P}\left(\mathcal{L}(x_n)_n \subset OZ\right) = 0.$$

Our second result states that the process will converge to attractors of the ODE (2) with positive probability.

Theorem 2 *Let A be an attractor of the ODE (2), with basin of attraction $\mathcal{B}(A)$. Then*

$$\mathbb{P}\left(\mathcal{L}(x_n)_n \subset A\right) > 0$$

on the event $\{x_0 \in \mathcal{B}(A)\}$.

This result does not rule out the possibility that the stochastic process will sometimes converge to a Nash equilibrium that is linearly unstable, nor does it rule out the possibility that it will sometimes converge to sets that are not even zeroes of the dynamics; but it guarantees that it will sometimes converge to stable equilibria, whenever they exist.

Taken together, theorems 1 and 2 tell us a lot and not much at the same time. On the one hand side, at this level of generality (almost no structure has been assumed on the game), it is surprising that such statements can be made. Natural candidates for convergence are the zeroes of the dynamical system. However, players will never converge to other zeroes, so among the set of zeroes, Nash equilibria survive as natural candidates for convergence. Furthermore, they also remain candidates for stability.

On the other hand, there are two drawbacks. First, at this level of generality we cannot exclude the possibility that the process converges to some set that is not in NE. Second, even when the process converges to the set of NE, there is no guarantee that it will converge to a stable equilibrium.

However, we can generally rule out convergence to interior[12] linearly unstable equilibria, under a simple topological property of the interaction network.

Definition 8 *A network is said to be bipartite if the set N of players can be partitioned into N_1 and N_2, such that for any pair of players i and j, we have*

$$g_{ij} = 1 \Longrightarrow (i \in N_1 \text{ and } j \in N_2) \text{ or } (i \in N_2 \text{ and } j \in N_1)$$

We will use the following well-known characterization of bipartite networks: A network is bipartite if and only if it contains no odd cycles.

Theorem 3 *Consider a network game with symmetric externalities and assume that the network is not bipartite. Let $\hat{x}$ be a linearly unstable interior Nash equilibrium. Then*

$$\mathbb{P}\left(\lim_n x_n = \hat{x}\right) = 0.$$

Again, this result is very general because the symmetric externalities assumption is satisfied by most of the games that are usually analyzed in economics and in particular in the social network literature. Then, as soon as the network is non bipartite, unstable equilibria can be ruled out.

Although the proof is detailed in the appendix, let us provide some intuition on this result. Our process is stochastic, because there is a random noise to it. Assume that at some period, the process goes near a stable equilibrium of the game. Then the deterministic dynamics (2) underlying the stochastic process (1) will drive it towards that stable equilibrium. However, the stochastic noise might push the process away, but because the equilibrium is stable, the dynamics will come back towards it.

[12] An equilibrium $x \in X$ is said to be interior if for every player, $x_i > 0$.

Now, assume the process goes near some unstable equilibrium. Again, the dynamics might drive the process towards that point. However, because the equilibrium is unstable, there is some direction (given by the eigenvector associated to the positive eigenvalues of the Jacobian matrix) such that, if the process follows it, it will go away from the equilibrium. However, in some circumstances the noise of the process might not allow it to take that very specific direction. We use the fact that when a network is not bipartite, it has an odd cycle. If in addition externalities are symmetric, then we show that the noise in the unstable direction never vanishes, so that that direction will be taken at some point in time.

Although non-bipartiteness and symmetric externalities are only sufficient conditions, we illustrate through two examples in the appendix why it is difficult to drop any of the two conditions.

Although this first set of result is very general, none rules out the possibility that the process converges to sets that are not in NE. We show in the next two sections that for broad classes of games, much more can be said.

4 Ordinal Potential Games

4.1 Definitions

Definition 9 *A game $\mathcal{G}$ is said to be a potential game if there is a function $P : X \to \mathbb{R}$ such that for all $x_{-i} \in X_{-i}$, for all $x_i, x_i' \in X_i$, we have*

$$u_i(x_i, x_{-i}) - u_i(x_i', x_{-i}) = P(x_i, x_{-i}) - P(x_i', x_{-i})$$

It is an ordinal potential game *(OPG) if P is differentiable and we have*[13]

$$sgn\left(\frac{\partial u_i}{\partial x_i}(x)\right) = sgn\left(\frac{\partial P}{\partial x_i}(x)\right) \tag{5}$$

Heuristically, this means that the variations in terms of payoffs for any agent is captured through a single function. In their paper, Monderer and Shapley (1996b) show that a continuous game with twice-differentiable payoffs is a potential game if and only if $\frac{\partial^2 u_i}{\partial x_i \partial x_j} = \frac{\partial^2 u_j}{\partial x_j \partial x_i}$ for all pair $i, j \in N$, with $i \neq j$.

We are not aware of any similar characterization for OPGs, but the class is larger, essentially because it allows for heterogeneity in interactions. This is the class we will focus on in this section.

We start by showing that most of the games considered in the network literature fall into this class. This allows the reader to get some feeling about the generality

[13] The original definition of (OPG) is $u_i(x_i, x_{-i}) - u_i(x_i', x_{-i}) > 0 \iff P(x_i, x_{-i}) - P(x_i', x_{-i}) > 0$, which is more restrictive when functions are differentiable.

of the results that will follow. The social network literature has intensely focused on games where actions are strategic substitutes or complements and the payoff of an agent depends on his own action and on the (possibly discounted) sum of the actions of his neighbors. We call these games *strict local aggregate games*. Formally,

Definition 10 *A game $\mathcal{G}$ defined on $X_i = [0, +\infty[$ is said to be a* strict local aggregate game *if, for any $i \in N$, there exists a twice differentiable map $v_i : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$, strictly concave in the first variable and such that*

(i) $u_i(x_i, x_{-i}) = v_i(x_i, \sum_{j \neq i} \delta_{ij} x_j)$*, where $\delta_{ij} \in \mathbb{R}$ and $\delta_{ij} = 0$;*

(ii) $\forall x \geq 0 \quad \lim_{x_i \to +\infty} \frac{\partial v_i}{\partial x_i}(x_i, y) < 0$*;*

(iii) either $\frac{\partial^2 v_i}{\partial x_i \partial y}(x_i, y) > 0$ *for all i, or* $\frac{\partial^2 v_i}{\partial x_i \partial y}(x_i, y) < 0$ *for all i.*

Obviously v_i inherits the concavity property of u_i: for any $x \geq 0$, $x_i \mapsto v_i(x_i, x)$ is strictly concave. Thus, strict local aggregate games are network games: let $x, x' \in X$ and $i \neq j$. Then we have

$$\frac{\partial u_i}{\partial x_j}(x) = \delta_{ij} \, \frac{\partial v_i}{\partial y} \left(x_i, \sum_{j \neq i} \delta_{ij} x_j \right).$$

Since $\frac{\partial^2 v_i}{\partial x_i \partial y}(x_i, y) \neq 0$, there exist a neighborhood U of x such that $\frac{\partial u_i}{\partial x_j} = 0$ on U if and only if $\delta_{ij} = 0$. The interaction graph is then given by $g_{ij} = 1 \Leftrightarrow \delta_{ij} \neq 0$. However, a strict local aggregate game might not exhibit symmetric externalities.

Moreover, for any $x \geq 0$, $x_i \mapsto v_i(x_i, x)$ admits a unique maximum on $\mathbb{R}_+$, which we will call the best response of player i against x by a slight abuse of language, and this best response is monotonic.

Many important economic applications fall into the category of local aggregate games. It is the case for instance of the Cournot competition presented in example 1. Here are other examples:

- Game of private provision of public goods on networks (Allouch, 2015, adapted from Bergstrom, Blume and Varian, 1986), with $X_i = [0, w_i]$, where $w_i > 0$ is player i's income, and $v_i(x_i, y) = h_i(w_i - x_i, x_i + y)$ where h_i is continuous, strictly increasing and strictly concave.

- Public good game on networks (Bramoullé and Kranton, 2007), with $X_i = [0, +\infty[$, $\delta_{ij} = \delta_{ji} \in \{0, \delta\}$ ($\delta \leq 1$), and $v_i(x_i, y) = b(x_i + y) - c.x_i$ where b is a strictly increasing concave function and $c > 0$ is a real number.

- Games with local strategic complements (Ballester-Calvó-Zenou, 2006), with $X_i = [0, +\infty[$, $\delta_{ij} \in \{0, \delta\}$ where $\delta \leq \lambda_{max}(G)$[14], and $v_i(x_i, y) = x_i - \frac{x_i^2}{2} + y x_i$

[14] $\lambda_{max}(G)$ denotes the highest eigenvalue of the adjacency matrix G. See the original paper for more details.

Notice that in strict local aggregate games, the best response maps b_i are strictly decreasing on $[0, x_0]$ for some $x_0 \in \overline{\mathbb{R}}_+$ in the case of substitutes, and strictly increasing on $[0, +\infty[$ in the case of complements. Let $H_i : \mathbb{R}^+ \to \mathbb{R}^+$ be defined by

$$H_i(x_i) = \int_0^{x_0} \min\{b_i(x), x_i\} dx$$

Then an easy adaptation of a more general result (see Theorem 1 in Dubey et al. (2006)) gives

Proposition 2 *Assume $\mathcal{G}$ is a strict local aggregate game, with $\delta_{ij} = \delta_{ji}$. Then $\mathcal{G}$ is an OPG, with potential P given by*

$$P(x) = \sum_{i \in N} \left(H_i(x_i) - \frac{1}{2} \sum_{j \neq i} \delta_{ij} x_i x_j \right)$$

The proof is in the appendix for convenience. For instance, the Cournot competition model of example 1 is a strict local aggregate game, with $\delta_{ij} = \delta_{ji} = 1$ if $g_{ij} = 1$, $\delta_{ij} = \delta_{ji} = 0$ otherwise. Proposition 2 tells us that it is an OPG and provides the potential function[15] :

$$P(x) = \sum_i \left(rx_i - \frac{x_i^2}{2} - \frac{1}{2} \sum_{j : g_{ij} = 1} x_i x_j \right); \tag{6}$$

Remark 5 *Obviously, not all OPGs are strict local aggregate games. Our results thus apply to a larger class than this one.*

4.2 Results on OPGs

Before we state our results on *OPG*s, we first define the following dynamical concept:

Definition 11 *Let $P : X \to \mathbb{R}$ be continuously differentiable. We say that P is a strict*[16] *Lyapunov function for $\dot{x} = G(x)$ if*

- *for $x \in Z(G)$ the map $t \mapsto P(\varphi(x, t))$ is constant;*
- *for $x \notin Z(G)$ the map $t \mapsto P(\varphi(x, t))$ is strictly increasing.*

[15] This potential can be found in Monderer and Shapley (1996b).

[16] Generally, P is a Lyapunov function for $\dot{x} = G(x)$ with respect to a component Λ if $t \mapsto P(\varphi(x, t))$ is constant on Λ and strictly increasing for $x \notin \Lambda$; when the component Λ coincides with the set of stationary points of the flow, then we say that P is strict.

Assume that $\mathcal{G}$ is an OPG with continuously differentiable potential P. Then P is a strict Lyapunov function for G (see Lemma 3 in appendix, where we also show that P is a strict Lyapunov function for the best-response dynamics $\dot{x} = -x + BR(x)$). In terms of convergence of our random process, the main feature of ordinal potential games is the following.

Theorem 4 *Assume that $\mathcal{G}$ is an OPG. Then*

a) If P is sufficiently regular, then

$$\mathbb{P}\left(\mathcal{L}(x_n)_n \subset Z(G)\right) = 1.$$

b) If $Z(G)$ consists in isolated points then

$$\mathbb{P}\left(\exists\, x^* \in NE : \ \lim_n x_n = x^*\right) = 1.$$

This result is a drastic improvement on theorems 1 and 2: the first statement guarantees that, for ordinal potential games, the only set to which our stochastic process can converge is the set of zeroes of G. Complex ω-limit sets of the dynamical system, that are non-zeroes, can be discarded. Combined with Theorem 1, we obtain the second statement, namely convergence to a Nash equilibrium when zeroes of G are isolated.

Given that many games in economic environments fall into the class under consideration, this result is particularly strong. Agents who play an OPG, of which they ignore everything except their own payoff, and who follow the simple behavior described here, will converge with probability one to the set of Nash equilibria, provided the zeroes of G are isolated.

We believe this result gives very strong foundations to the use of Nash equilibria in many economic contexts when the strategy space is continuous. Indeed, if Nash play is difficult to assume in discrete games, because the convergence of learning processes happens only in limited circumstances, this result shows that it is a much more natural concept in the case of continuous games.

As for stability, here also we can say much more, since we know how to distinguish between stable and unstable sets, by relating attractors of the dynamics to the potential function P.

Definition 12 *Let G be a smooth map, Λ be a compact connected subset of $Z(G)$ and let $N^\delta(\Lambda) := \{y \in X : d(y, \Lambda) < \delta\}$. We say that Λ is an* isolated component *of $Z(G)$ if there exists $\delta > 0$ such that $N^\delta(\Lambda) \cap Z(G) = \Lambda$.*

Notice that an isolated component extends the notion of isolated points that we have referred to earlier.

Definition 13 *Let G be a smooth map and Λ be an isolated component of $Z(G)$, we say that Λ is a local maximum of P if*
(i) P is constant on Λ: $P(x) = v \ \forall x \in \Lambda$;
(ii) there exists an open neighborhood U of Λ such that $P(y) \leq v \ \forall y \in U$

Theorem 5 *Assume $\mathcal{G}$ is an OPG and let Λ be an isolated component of $Z(G)$. Then the following statements are equivalent*

(i) Λ is an attractor for $\dot{x} = G(x)$

(ii) Λ is a local maximum of P.

(iii) $\Lambda \subset NE$ and Λ is an attractor for the best response dynamics $\dot{x} = -x + BR(x)$.

We illustrate with example 1, on the network in Figure 1. Then, although the set $\{(0, \alpha, 0, r-\alpha); \alpha \in [0, r]\}$ is an isolated component in the set of Nash equilibria, it is not an isolated component of $Z(G)$. Indeed, the corresponding isolated component in $Z(G)$ is $\Lambda = \{(\beta, \alpha, \delta, r-\alpha-\beta-\delta); \alpha, \beta, \delta \in [0, r], \beta\delta = 0$ and $r-\alpha-\beta-\delta \geq 0\}$. The potential (6) is constant and equal to $r^2/2$ along this component, of which the profile $(0, 0, 0, r)$ is a member. However, for small enough $\epsilon > 0$, we have $P(\epsilon, 0, \epsilon, r - 2\epsilon) = \frac{r^2}{2} + \epsilon^2$, with $(\epsilon, 0, \epsilon, r - 2\epsilon) \notin \Lambda$. Theorem 5 then tells us that Λ is not an attractor.

Put together, theorems 4 and 5 are very informative and positive: when zeroes of G are isolated then the process almost surely converges to a Nash equilibrium. However, at this stage there is no guarantee that the process cannot converge to an unstable equilibrium. On the other hand, when $Z(G)$ contains continuum of equilibria then Theorem 4, b) is not relevant and part a) gives us strong, yet partial information on the limit set of $(x_n)_n$. Theorem 2, however, tells us that attractors are special sets because they contain the limit set of $(x_n)_n$ with positive probability. Identifying attractors is therefore of prime importance, and this is what Theorem 5 does. It states that the set of attractors coincides with the set of local maxima of the potential.

As we stated earlier, we obtain some information about the Best-Response dynamics, as a by-product of our analysis. In the case of OPGs, this information is particularly precise: first, attractors are necessarily included in the set of Nash equilibria, so that other zeroes are irrelevant; second, attractors also coincide with the local maxima of P.

5 Games with strategic complements

The previous class contains some games with strategic complements, but obviously not all of them. Nevertheless, games with strategic complements have well structured sets of Nash equilibria. This helps us in finding the limit set of process.

Under the assumptions we have made (i.e. best-responses are unique and payoff functions are strictly concave), it is easy to check that the best response functions are differentiable and strictly increasing. In that case, Vives (1990, Theorem 5.1) proved that for a specific set of initial conditions[17], the discrete-time best response dynamics monotonically converges to an equilibrium point. For the continuous-time best response dynamics, Vives (1990, Remark 5.2) observes that the same results holds for almost all initial conditions x_0 in the state space.

This is a useful result when one analyses the deterministic system $\dot{x} = G(x)$ underlying our stochastic process, because as we have seen, it shares some properties with the best-response dynamics $\dot{x} = -x + BR(x)$. Unfortunately, this does not really help in understanding the behavior of the stochastic process. Indeed, the set of problematic initial conditions cannot be discarded. In fact, it could be that the stochastic process passes through these points very often, in which case the above result is ineffective. Moreover, it is known[18] that the dynamical system, once it passes through one of these points, can converge to very complicated sets, called repulsive normally hyperbolic sets. As we see, we cannot rely on any of these existing results. However, we still get very precise statements.

In order to get our results, we need to make sure that our dynamics satisfy some mild conditions. We first detail these conditions. Let $DG(x)$ denote the Jacobian matrix of the dynamical system:

$$DG(x) = \begin{bmatrix} x_1 \frac{\partial^2 u_1}{\partial x_1^2}(x) + \frac{\partial u_1}{\partial x_1}(x) & x_1 \frac{\partial^2 u_1}{\partial x_1 \partial x_2}(x) & \ldots & x_1 \frac{\partial^2 u_1}{\partial x_1 \partial x_n}(x) \\ x_2 \frac{\partial^2 u_2}{\partial x_2 \partial x_1}(x) & x_2 \frac{\partial^2 u_2}{\partial x_2^2}(x) + \frac{\partial u_2}{\partial x_2}(x) & \ldots & x_2 \frac{\partial^2 u_2}{\partial x_2 \partial x_n}(x) \\ \ldots & \ldots & \ldots & \ldots \\ x_n \frac{\partial^2 u_n}{\partial x_n \partial x_1}(x) & \ldots & x_n \frac{\partial^2 u_n}{\partial x_n \partial x_{n-1}}(x) & x_n \frac{\partial^2 u_n}{\partial x_n^2}(x) + \frac{\partial u_n}{\partial x_n}(x) \end{bmatrix}$$

The dynamics $\dot{x} = G(x)$ is said to be *cooperative* and *irreducible* in x if the Jacobian matrix $DG(x)$ has nonnegative off-diagonal entries and is irreducible. Because dynamics sharing these properties have nice features[19], we will impose some conditions which guarantee that, for games with strategic complements, DG always has non-negative off-diagonal entries. However it is obviously not irreducible on the boundary of the state space. Therefore we need to make sure that the boundary is repulsive and the Jacobian matrix is irreducible in the interior of the state space.

Hypothesis 2 *For any agent i,*

$$\frac{\partial u_i}{\partial x_i}(0,0) > 0.$$

[17]More specifically, the initial condition x_0 must be such that either $x_0^i \geq BR_i(x_0^{-i})$ for all i or $x_0^i \leq BR_i(x_0^{-i})$ for all i.

[18]See Theorem 8 in the Appendix.

[19]The vector field G being cooperative and irreducible, the associated flow has positive derivatives: $\frac{\partial \varphi(x,t)}{\partial x} \gg 0$ for $t > 0$. This implies that it is *strongly monotonic*: $\varphi(x,t) \gg \varphi(y,t)$ for all $x > y$ and $t > 0$.

Hypothesis 3 *The matrix $DG(x)$ is irreducible for any x in the interior of X*

These two assumptions are quite natural. In the case of network games, the later more or less means that the interaction graph is connected. The former, in turn, guarantees that the boundary contains no Nash equilibrium, so that they all lie in the interior of the state space. However, the boundary still contains the other zeroes and we would like to know that the process $(x_n)_n$ does not get there. We show that this cannot happen, because the boundary is repulsive:

Theorem 6 *Under Hypothesis 2, there exists $a > 0$ such that $\mathcal{L}((x_n)_n) \subset [a, +\infty[^N$ almost surely.*

Theorem 6 is the counterpart of Theorem 1 in section 3, which guarantees that the limit set is not included in other zeroes. Here it is a stronger result because it proves that the process will not even go close to those: it will be away from the boundary. Therefore all the action will take place in the interior of the state space, where Hypothesis 3 holds.

Next, we show the counterpart of theorem 2, but we get a much stronger result: under some condition on the noise component U_{n+1}, we can guarantee that not only the process will converge to Nash, but it will converge to a stable Nash equilibrium, at least for non-bipartite graphs. Our condition, stated in the next definition, assumes that the noise is "exciting" in every direction in the interior of the state space. This will guarantee that that the process will not converge to general unstable sets[20]. Of course, since we know that the process approaches the boundary with null probability, we only need to take care of what might happen in the interior.

Definition 14 *Let $Q(x) := \mathbb{E}(U_{n+1}^T U_{n+1} \mid x_n = x)$. We say that the noise is uniform if $Q(x)$ is positive definite for any $x \in Int(X) = (\mathbb{R}_+^*)^N$*

Let us recall that a matrix is positive definite if all its eigenvalues are strictly positive. This condition naturally implies that the noise can go in every direction.

Remark 6 *As we will see in the appendix, if the game has symmetric externalities, then the noise is uniform.*

This allows us to state the following very general theorem: Let $Z^U(G)$ denote the set of linearly unstable equilibria and $Z^S(G) = Z(G) \setminus Z^U(G)$ is the set of stable equilibria.

Theorem 7 *Consider a network game with strategic complements and real analytic payoff functions. Assume that Hypothesis 2 and 3 hold, and that the game exhibits*

[20]Linearly unstable equilibria are unstable sets, but unstable sets also include much more complex structures. Excluding linearly unstable sets is not difficult, as we will see next, but excluding other sets is tricky.

symmetric externalities, on a non-bipartite graph. Then our learning process (x_n) *almost surely converges to a stable Nash equilibrium:*

$$\mathbb{P}\left(\exists\, x^* \in Z^S(G) : \ \lim_n x_n = x^*\right) = 1$$

on the event $\{\limsup_n \|x_n\| < +\infty\}$*.*

This result is very strong, in particular because the assumptions are verified for most economic models we have in mind, and because they are weak in comparison to what we have assumed on individuals' knowledge of the game. Obviously, on the event that $(x_n)_n$ is unbounded, there is nothing to say. For instance for the classical linear-quadratic payoffs with strategic complements introduced in Ballester et al. (2006), when the interaction term is too large and there is no Nash equilibrium, (x_n) will clearly go to infinity and our analysis is irrelevant. A simple sufficient condition that guarantees that $\mathbb{P}(\limsup_n \|x_n\| < +\infty) = 1$ is that the dynamics $\dot{x} = G(x)$ admits a global attractor (which is the case in the linear-quadratic model when interactions are small enough).

Remark 7 *Notice that here again we need a condition on the non-bipartiteness of the graph. This is again because on these specific structures the noise can vanish in some directions. In the appendix, after the proof of theorem 7, we illustrate this point with a specific game. If we are to relax the assumption on non-bipartiteness, then we still get a positive result that we prove in the appendix:*
Consider a game (not necessarily a network game) that exhibits strategic complements and symmetric externalities. Let $\hat{x}$ *be a linearly unstable interior zero of* G*. Then*

$$\mathbb{P}\left(\lim_n x_n = \hat{x}\right) = 0.$$

6 Conclusion

In this paper we have addressed a simple question, that naturally follows from the growth of the social network literature. Given the difficulty to get convergence to Nash equilibria in standard games, is there any chance that agents, playing on a network, ever converge to anything? We examined the case of continuous games, because these are the games that have most been analyzed in the social network literature, and we have assumed the simplest possible sophistication of agents: they do not know anything. Theu only observe their payoff at the end of each round and adapt their next choice according to how their payoff has varied.

With this simple framework, we prove many positive results that show that Nash equilibrium is a very natural concept in the context of continuous games. They are candidates for convergence in any concave game, and they will often be the only possible converging points in OPGs and in games with complementarities. Furthermore, in the first case, stable equilibria are likely to emerge, while in the second case, only stable Nash equilibria will emerge.

Appendix

6.1 Proof of results of Section 2

Proof of Lemma 1 We have, for any $i \in N$,

$$e^i_{2n+2} - e^i_{2n} = e^i_{2n}\epsilon^i_n \Delta u^i_{n+1}$$

A first order development gives

$$\begin{aligned}\epsilon^i_n \Delta u^i_{n+1} &= \epsilon^i_n \left(u_i \left(e^i_{2n} + \frac{1}{n+1}\epsilon^i_n, e^{-i}_{2n} + \frac{1}{n+1}\epsilon^{-i}_n \right) - u_i(e^i_{2n}, e^{-i}_{2n}) \right) \\ &= \frac{1}{n+1}(\epsilon^i_n)^2 \frac{\partial u_i}{\partial x^i}(e_{2n}) + \frac{1}{n+1}\epsilon^i_n \sum_{j \neq i} \epsilon^j_n \frac{\partial u_i}{\partial x^j}(e_{2n}) + \mathcal{O}(\frac{1}{n^2})\end{aligned}$$

Because $(\epsilon^i_n)^2 = 1$ and $x_n = e_{2n}$, we have

$$x^i_{n+1} - x^i_n = \frac{1}{n+1} x^i_n \frac{\partial u_i}{\partial x^i}(x_n) + \frac{1}{n+1}\epsilon^i_n x^i_n \sum_{j \neq i} \epsilon^j_n \frac{\partial u_i}{\partial x^j}(x_n) + \mathcal{O}(\frac{1}{n^2})$$

By setting $U^i_{n+1} = \epsilon^i_n x^i_n \sum_{j \neq i} \epsilon^j_n \frac{\partial u_i}{\partial x^j}(x_n)$, we get equation (1). Finally, note that $\mathbb{E}(\epsilon^j_n) = 0$ for all j, and that ϵ^i_n and ϵ^j_n are independent, so that

$$\mathbb{E}(U_{n+1} \mid \mathcal{F}_n) = 0 \quad \blacksquare$$

Next, we prove that the process is well-defined, in the sense that actions stay positive at each step.

Lemma 2 *For $n \geq 0$, we have $x^i_n > \frac{1}{n+1}$ and $e^i_{2n+1} > 0$, provided $x^i_0 > 1$.*

Proof. Let $n \geq 0$. By assumption, $|u_i(x) - u_i(x')| \leq \|x - x'\|_\infty$ for all i. Thus,

$$\frac{x^i_{n+1}}{x^i_n} \geq (1 - \|e_{2n+1} - x_n\|_\infty)$$

and $|e^i_{2n+1} - x^i_n| \leq \frac{1}{n+2}$ for all i. As a consequence

$$\frac{x^i_{n+1}}{x^i_n} \geq (1 - \frac{1}{n+2}).$$

and

$$x^i_n \geq x^i_0 \prod_{k=0}^{n-1} \left(1 - \frac{1}{k+2}\right) = \frac{1}{n+1} x^i_0 > \frac{1}{n+1}.$$

Moreover $e^i_{2n+1} \geq x^i_n - \frac{1}{n+2} > 0. \quad \blacksquare$

6.2 Proof of results of Section 3

Proof of Remark 4 Let $x \in Z(G) \setminus NE$ and assume without loss of generality that $x_1 = 0$, $\frac{\partial u_1}{\partial x_1}(x) > 0$. Then it is not hard to see that the first column of DG, evaluated in x is $(\frac{\partial u_1}{\partial x_1}(x), 0, ..., 0)^T$. ■

Proof of Theorems 1 and 6.

• Let us first prove Theorem 1. observe first that we can work on the event $\left\{\sup_{n\to+\infty} \|x_n\| < +\infty\right\}$ since, otherwise, there is nothing to prove.

For $\eta > 0$, let $K_\eta \subset Z(G) \setminus NE$ be the set defined by

$$K_\eta = \left\{ x \in X: \; x^j \frac{\partial u^j}{\partial x^j} = 0 \; \forall\, j \in \{1, \ldots, N\} \text{ and } \exists\, i \text{ with } x^i = 0 \text{ and } \frac{\partial u^i}{\partial x^i} \geq 2\eta \right\},$$

Clearly, $(K_\eta)_{\eta>0}$ is a collection of compact sets,that is increasing as η decreases to zero, and $Z(G) \setminus NE = \cup_{\eta>0 \in \mathbb{N}^*} K_\eta$. Thus the family of events $E_\eta = \{\mathcal{L}((x_n)_n) \subset K_\eta\}$ is increasing, as η decreases to zero, and $\cup_{\eta>0} E_\eta = \{\mathcal{L}((x_n)_n) \subset Z(G) \setminus NE\}$, since $\mathcal{L}((x_n)_n)$ is compact. Assume by contradiction that $\mathbb{P}\left(\mathcal{L}((x_n)_n) \subset Z(G) \setminus NE\right) > 0$. Then necessarily there exists $\eta^* > 0$ such that $\mathbb{P}\left(E_{\eta^*}\right) > 0$. On this event, there exists $n^* \in \mathbb{N}$ (random) such that $\frac{\partial u^i}{\partial x^i}(x_n) \geq \eta^*$, $\forall n \geq n^*$.

Let $\tilde{U}^i_{n+1} = \epsilon^i_n \sum_j \epsilon^j_n \frac{\partial u_i}{\partial x_j}(x_n)$, so that $x^i_{n+1} = x^i_n \left(1 + \frac{1}{n+1}\left(\frac{\partial u_i}{\partial x_i}(x_n) + \tilde{U}^i_{n+1} + \frac{\xi^i_{n+1}}{x^i_n}\right)\right)$. Define the random variable $y_n = \prod_{i=1}^N x^i_n \geq 0$. We then have that

$$\begin{aligned} y_{n+1} &= \prod_{i=1}^N x^i_n \left(1 + \frac{1}{n+1}\left(\frac{\partial u_i}{\partial x_i}(x_n) + \tilde{U}^i_{n+1} + \frac{\xi^i_{n+1}}{x^i_n}\right)\right) \\ &= y_n \left(1 + \sum_{i=1}^N \frac{1}{n+1}\left(\frac{\partial u_i}{\partial x_i}(x_n) + \tilde{U}^i_{n+1}\right) + \frac{\xi^i_{n+1}}{x^i_n}\right) \end{aligned}$$

As a consequence, since $\xi^i_n = \mathcal{O}(\frac{1}{n^2})$ and $x^i_n \geq 1/(n+1)$, we have

$$\frac{1}{y_{n+1}} = \frac{1}{y_n}\left(1 - \sum_{i=1}^N \frac{1}{n+1}\left(\frac{\partial u_i}{\partial x_i}(x_n) + \tilde{U}^i_{n+1}\right) + o(\frac{1}{n})\right).$$

For $n \geq n^*$, we have $\sum_{i=1}^N \frac{1}{n+1}\frac{\partial u_i}{\partial x_i}(x_n) \geq \eta^* \frac{1}{n+1}$, which implies that

$$\frac{1}{y_{n+1}} \leq \frac{1}{y_n} - \frac{1}{y_n}\frac{\eta^*}{2}\frac{1}{n+1} - \frac{1}{y_n}\sum_{i=1}^N \tilde{U}^i_{n+1}.$$

As a consequence

$$\mathbb{E}\left(\frac{1}{y_{n+1}} \mid \mathcal{F}_n\right) \leq \frac{1}{y_n} - \frac{1}{y_n}\frac{\eta^*}{2}\frac{1}{n+1},$$

that is, the random sequence $(1/y_n)_n$ is a positive supermartingale. It then converges almost surely to some random variable Y. However, on the event $\{\mathcal{L}((x_n)_n) \subset Z(G) \setminus NE\}$, we have almost surely $y_n \to_n 0$. these two convergence properties are in contradiction. Thus $\mathbb{P}\left(\mathcal{L}((x_n)_n) \subset Z(G) \setminus NE\right) = 0$.

• The proof of Theorem 6 is in the same spirit. Under assumption 2, for any i, there exists $\overline{x}_i > 0$ such that

$$\frac{\partial u_i}{\partial x_i}(x_i, 0) > \alpha_i > 0, \; \forall x_i < \overline{x}_i.$$

Since the game has strategic complements,

$$\frac{\partial u_i}{\partial x_i}(x_i, x_{-i}) > \alpha_i > 0, \; \forall x_i < \overline{x}_i, \; \forall x_{-i} \in X_{-i}.$$

As a consequence, any solution trajectory with initial condition in the set $\{x \in X : x_i \in]0, \overline{x}_i[\}$ is in the set $\{x \in X : x_i > \overline{x}_i\}$ after some time $t > 0$. Let $a = \max_i \overline{x}_i$. Then either $\mathcal{L}((x_n)_n) \subset [a, \infty[^N$ or $\mathcal{L}((x_n)_n) \subset \partial X$.

Let $\eta := \min_{i=1,\ldots,N} \frac{\partial u_i}{\partial x_i}(0,0) > 0$. Clearly

$$\partial X \subset \{x \in X : \; \exists i \text{ with } \; x_i = 0 \text{ and } \frac{\partial u_i}{\partial x_i}(x) \geq \eta\}$$

The end of the proof follows the same argument as above. ■

Proof of Theorem 2 This result can be found in Benaïm (1999).

Proof of Theorem 3 We assume without loss of generality that the unstable space at $\hat{x}$ is one-dimensional, that is $DG(\hat{x})$ has only one strictly positive eigenvalue μ, and we call v the associated normalized eigenvector. We use a result from Pemantle (1990), more precisely in the settings of Brandiere and Duflo (1996), which gives the following sufficient condition for non convergence $\hat{x}$:

$$\liminf_n \mathbb{E}\left(| < U_{n+1}, v > |^2 \mid \mathcal{F}_n\right) > 0 \text{ on the event } \left\{\lim_n x_n = \hat{x}\right\}. \tag{7}$$

We show that under our assumptions, this conditions is always met.

We have

$$< U_{n+1}, v >^2 = \left(\sum_{i<j} \epsilon_n^i \epsilon_n^j \left(v_i x_n^i \frac{\partial u_i}{\partial x^j}(x_n) + v_j x_n^j \frac{\partial u_j}{\partial x^i}(x_n)\right)\right)^2$$

Hence, using $\mathbb{E}\left(\epsilon_n^i \epsilon_n^j\right) = 0$ if $i \neq j$ and $(\epsilon_n^i)^2 = 1$, we get

$$\mathbb{E}\left(< U_{n+1}, v >^2 \mid \mathcal{F}_n\right) \;\; = \;\; \sum_{i<j} \left(v_i x_n^i \frac{\partial u_i}{\partial x^j}(x_n) + v_j x_n^j \frac{\partial u_j}{\partial x^i}(x_n)\right)^2$$

and

$$\lim_{n\to+\infty} \mathbb{E}\left(< U_{n+1}, v >^2 | \ \mathcal{F}_n\right) = \sum_{i<j} \left(v_i \hat{x}^i \frac{\partial u_i}{\partial x^j}(\hat{x}) + v_j \hat{x}^j \frac{\partial u_j}{\partial x^i}(\hat{x}) \right)^2$$

on the event $\{\lim_n x_n = \hat{x}\}$. Hence, the condition is verified in v if and only if

$$\forall i < j, \ \ v_i \hat{x}^i \frac{\partial u_i}{\partial x^j}(\hat{x}) + v_j \hat{x}^j \frac{\partial u_j}{\partial x^i}(\hat{x}) = 0.$$

We now prove that under the assumption of symmetric externalities and non-bipartiteness of the graph, this quantity is positive.

Since the interaction graph is non bipartite in $\hat{x}$, there is at least one odd cycle. Let us assume, for simplicity but without loss of generality, that this cycle is of length 3: there exists i, j, k such that

$$\frac{\partial u_i}{\partial x_j}(\hat{x}) \frac{\partial u_j}{\partial x_i}(\hat{x}) > 0, \frac{\partial u_j}{\partial x_k}(\hat{x}) \frac{\partial u_k}{\partial x_j}(\hat{x}) > 0, \frac{\partial u_i}{\partial x_k}(\hat{x}) \frac{\partial u_k}{\partial x_i}(\hat{x}) > 0.$$

We thus have $sgn(v_i) = -sgn(v_j) = sgn(v_k) = -sgn(v_i)$ which implies, since $\hat{x}$ is interior, that $v_i = v_j = v_k = 0$. As a consequence, for every agent l linked to i, j or k, we have $v_l = 0$. Recursively, since the interaction graph is connected, we have $v = 0$, which is a contradiction and concludes the proof. ■

Remark 8 *At this point, some comments are in order. The key argument that is needed to rule out convergence to linearly unstable equilibria is the noise condition (7). When $\hat{x}$ is interior and linearly unstable, it is clearly not necessary to assume non-bipartiteness of the network and/or symmetric externalities for this condition to hold. However, as we shall see in the next example, we might run into troubles in bipartite graphs if the game does not have strategic complements. We provide an example with strategic substitutes with 4 agents on a square. Consider the following 4-player public good game:*

$$u_1(x) = -cx_1 + b(x_1 + x_2 + x_4), u_2(x) = -cx_2 + b(x_2 + x_1 + x_3),$$

$$u_3(x) = -cx_3 + b(x_3 + x_2 + x_4), u_4(x) = -cx_4 + b(x_4 + x_1 + x_3),$$

with b strictly concave and such that $b'(1) = c$. It is not hard to see that $\hat{x} = (1/3, 1/3, 1/3, 1/3)$ is a Nash equilibrium. Choosing b such that $b''(1) = -3$ for simplicity, the Jacobian matrix associated to $\hat{x}$ is

$$DG(\hat{x}) = \begin{pmatrix} -1 & -1 & 0 & -1 \\ -1 & -1 & -1 & 0 \\ 0 & -1 & -1 & -1 \\ -1 & 0 & -1 & -1 \end{pmatrix},$$

which corresponds to a saddle point, as the eigenvalues are $-3, -1, -1, 1$. *The eigenspace associated to the positive eigenvalue is generated by* $v = (1, -1, 1, -1)$ *so that, on the event* $\{\lim_n x_n = \hat{x}\}$, *we have*

$$\begin{aligned}\lim_{n\to+\infty} \mathbb{E}\left(< U_{n+1}, v >^2 \middle| \mathcal{F}_n\right) &= \left(\frac{\partial u_1}{\partial x_2}(\hat{x}) - \frac{\partial u_2}{\partial x_1}(\hat{x})\right)^2 \\ &+ \left(-\frac{\partial u_2}{\partial x_3}(\hat{x}) + \frac{\partial u_3}{\partial x_2}(\hat{x})\right)^2 + \left(\frac{\partial u_3}{\partial x_4}(\hat{x}) - \frac{\partial u_4}{\partial x_3}(\hat{x})\right)^2 = 0\end{aligned}$$

and the noise condition here does not hold.

6.3 Proof of results of Section 4

Proof of Proposition 2. Without loss of generality, we write the proof for the substitutes case. First remark that $b_i : [0, x_0[\to]0, b_i(0)]$ admits an inverse function $b_i^{-1} :]0, b_i(0)] \to [0, x_0[$, which is continuously differentiable and strictly decreasing. We extend b_i^{-1} through $b_i^{-1}(x_i) = (b_i^{-1})'(b_i(0))(x_i - b_i(0))$ for $x_i \geq b_i(0)$. Note that b_i^{-1} is then strictly decreasing on $\mathbb{R}_+$. By straightforward computations, we have

$$H_i(x_i) = x_i b_i^{-1}(x_i) + \int_{b_i^{-1}(x_i)}^{x_0} b_i(x)dx.$$

Hence we have

$$H_i'(x_i) = b_i^{-1}(x_i).$$

Consequently

$$\langle DP(x), G(x)\rangle = \sum_{i\in N} x_i \frac{\partial u_i}{\partial x_i}(x)\left(b_i^{-1}(x_i) - \sum_{j\in N}\delta_{i,j}x_j\right).$$

$$\frac{\partial u_i}{\partial x_i}(x_i, x) > 0 \iff x_i < b_i(x) \iff b_i^{-1}(x_i) > x \iff \frac{\partial P}{\partial x_i}(x_i, x) > 0.$$

and

$$\frac{\partial u_i}{\partial x_i}(x_i, x) < 0 \iff x_i < b_i(x) \iff b_i^{-1}(x_i) < x \iff \frac{\partial P}{\partial x_i}(x_i, x) < 0.$$

This concludes the proof. ■

Lemma 3 *Assume that* $\mathcal{G}$ *is an OPG with continuously differentiable potential* P. *Then*

(*i*) P *is a Lyapunov function for* $\dot{x} = -x + BR(x)$, *with respect to* NE.

(ii) *P is a Lyapunov function for* $\dot{x} = G(x)$ *(where* $G_i(x) = x_i \frac{\partial u_i}{\partial x_i}(x)$*) with respect to* $Z(G)$.

Proof. By assumption,

$$\forall x, \forall i, \ \frac{\partial u_i}{\partial x_i}(x) > 0 \Rightarrow \frac{\partial P}{\partial x_i}(x) > 0 \text{ and } \frac{\partial u_i}{\partial x_i}(x) < 0 \Rightarrow \frac{\partial P}{\partial x_i}(x) < 0.$$

(i) We have

$$\langle DP(x), -x + BR(x) \rangle = \sum_i \frac{\partial P}{\partial x_i}(x)(-x_i + BR_i(x)).$$

We need to check that, if $x \notin NE$, then this quantity is positive. Let i be such that $x_i \neq BR_i(x)$, say $x_i < BR_i(x_{-i})$. Then by strict concavity of u_i we have $\frac{\partial u_i}{\partial x_i}(x) > 0$. Thus $\frac{\partial P}{\partial x_i}(x) > 0$ and $\langle DP(x), -x + BR(x) \rangle > 0$.

(ii) We have

$$\langle DP(x), G(x) \rangle = \sum_i x_i \frac{\partial u_i}{\partial x_i}(x) \frac{\partial P}{\partial x_i}(x).$$

We need to check that, if $x \notin Z(G)$, then this quantity is positive. Let i be such that $G_i(x) \neq 0$. Then $x_i > 0$ and $\frac{\partial u_i}{\partial x_i}(x) \neq 0$, which implies that

$$x_i \frac{\partial u_i}{\partial x_i}(x) \frac{\partial P}{\partial x_i}(x) > 0.$$

and the proof is complete. ■

Proof of Theorem 4.

Part a): The first point is a consequence of the Proposition 6.4 in Benaïm (1999). We prove that we can use this theorem:

Lemma 4 *Assume $\mathcal{G}$ is an OPG. Then, if P is sufficiently regular, $P(Z(G))$ has an empty interior.*

Proof of Lemma 4. The sketch of this proof is as follows: we decompose the set of zeroes of G as a finite union of sets on which we can use Sard's Theorem.

Let A be any subset of agents and Z_A be the set

$$\left\{ x \in Z(G) : \ x_i = 0 \ \forall i \notin A, \ \frac{\partial u_i}{\partial x_i} = 0 \ \forall i \in A \right\}.$$

It is not hard to see that Z_A is closed. Moreover $Z(G) = \cup_{A \in \mathcal{P}(\{1,\ldots,N\})} Z_A$.

We now prove that P is constant on Z_A. Let $P^A : [0,1]^A \to \mathbb{R}$ be defined as

$$P^A(z) := P(z, 0).$$

For $x \in Z_A$, denote by $x^A = (x_i)_{i \in A}$. We then have $P^A(x^A) = P(x)$. Moreover, for $i \in A$,

$$\frac{\partial P^A}{\partial x_i} = 0$$

by definition of Z_A and the additional assumption we made on P. Hence

$$\{x^A : \ x \in Z_A\} \subset \{z \in [0,1]^A : \ \nabla_z P^A = 0\}.$$

Now P is sufficiently differenciable, so is P^A, and by Sard's Theorem, $P^A(\{x^A : \ x \in Z_A\})$ has empty interior in $\mathbb{R}^A$. As an immediate consequence, P^A is constant on $\{x^A : \ x \in Z_A\}$, which directly implies that $P(Z_A)$ has empty interior. Since $Z(G)$ is a finite union of such sets, $P(Z(G))$ has empty interior. ■

Part *b*): The second point follows directly from Proposition 6.6 in Benaïm (1999) and Theorem 1. ■

Proof of Theorem 5. First we prove that (i) implies (ii). Since $\mathcal{G}$ is an OPG, $P(Z(G))$ has empty interior (see lemma 4 above). Moreover, by virtue of theorem 4, we have $\Lambda \subset Z(G)$. Then P is constant on Λ. Let $v := P(\Lambda)$. If Λ is not a local maximum of P then there exists a sequence x_n such that $d(x_n, \Lambda) \to_n 0$ and $P(x_n) > v$. Since Λ is isolated we have $x_n \in X \setminus Z(G)$ and $P(\varphi(x_n, t)) > P(x_n) > v$ for any $t > 0$ hence $d(\varphi(x_n, t), \Lambda) \nrightarrow 0$ and Λ is not an attractor.

Let us now prove that (ii) implies (iii). First we show that Λ is contained in NE. Suppose that there exists $\hat{x} \in \Lambda \setminus NE$. Without loss of generality, we suppose that

$$\hat{x}_1 = 0, \ \frac{\partial u_1}{\partial x_1}(\hat{x}) > 0$$

Since $\frac{\partial u_1}{\partial x_1}(\hat{x}) > 0$, we also have $\frac{\partial P}{\partial x_1}(\hat{x}) > 0$, by definition of an OPG. As a consequence, $\hat{x}$ is not a local maximum of P.

We now prove that Λ is an attractor for the best response dynamics. P is a strict Lyapunov function for the best response dynamics[21] and $\Lambda \subset NE$. The statement we want to prove is then a consequence of Proposition 3.25 in Benaïm et al. (2005) (maybe an older reference for ode?) We adapt the proof in our context for convenience. First of all observe that Λ is actually a strict local maximum of P: there exists an open (isolating) neighborhood U of Λ such that $P(x) < v = P(\Lambda), \ \forall x \in U \setminus \Lambda$. This is a simple consequence of the fact that P is strictly increasing along any solution curve with initial conditions in $U \setminus \Lambda$. .Now let $V_r := \{x \in U : P(x) > v - r\}$. Clearly $\cap_r V_r = \Lambda$.

[21] Keep in mind that this means that it is a lyapunov function with respect to NE.

Also $\varphi(\overline{V_r}, t) \subset V_r$, for $t > 0$, r small enough[22]. This implies that $\Lambda = \cap_{r>0} V_r$ contains an attractor A. The potential being constant on Λ, A cannot be strictly contained in Λ and therefore Λ is an attractor.

Now clearly (iii) implies (i): $\Lambda = \omega_{Br}(U)$ for some open neighborhood U of Λ. Since $U \cap Z(G) \subset NE$, $\omega_G(U) = \omega_{Br}(U)$ and the proof is complete ■

6.4 Proof of results of Section 5

Proof of Theorem 6 The proof can be found with the proof of Theorem 1 above.

Proof of Remark 7 We assume without loss of generality that the unstable space at $\hat{x}$ is one-dimensional and generated by the unitary vector v, corresponding to the positive eigenvalue μ, and that the interaction graph is connected bipartite (the other case is covered by Theorem 3). Then there exists a partition of N, (A, B), such that if $a \in A$ and $\frac{\partial u_a}{\partial x_b}(\hat{x})\frac{\partial u_b}{\partial x_a}(\hat{x}) > 0$ then $b \in B$. Using the computations developed in the proof of Theorem 3, we need to show that

$$\sum_{a<b} \left(v_a x^a \frac{\partial u_a}{\partial x^b}(\hat{x}) + v_b x^b \frac{\partial u_a}{\partial x^b}(\hat{x}) \right)^2 \neq 0.$$

Assume the contrary. Then, since the interaction graph is connected, without loss of generality we may assume that $v_a > 0\ \forall a \in A$ and $v_b < 0\ \forall b \in B$. By a simple rearrangement of the indexes, the Jacobian matrix in $\hat{x}$ can be written as follows:

$$DG(\hat{x}) = \begin{pmatrix} D_A & M \\ N & D_B \end{pmatrix},$$

where D_A is diagonal and the diagonal terms are equal to $(x^a \partial^2 u_a / \partial (x^a)^2)(\hat{x}) \leq 0$, $a \in A$; and similarly for D_B. M and N are non-negative matrices, as $x^i \partial^2 u_i / \partial x^i \partial x^j \geq 0\ \forall i \neq j$. Because μ is strictly positive and v is the corresponding normalized eigenvector, we should have $\langle vDG(\hat{x}), v \rangle = \mu \sum_i v_i^2 > 0$, since $v \neq 0$. However,

$$\langle vDG(\hat{x}), v \rangle = \sum_i v_i^2 x^i \partial^2 u_i / \partial (x^i)^2 + \sum_{a\in A, b\in B} v_a v_b \left(x^a \frac{\partial^2 u_a}{\partial x^a \partial x^b} + x^b \frac{\partial^2 u_b}{\partial x^a \partial x^b} \right) \leq 0,$$

a contradiction. ■

Remark 9 *If externalities are not symmetric in $\hat{x}$ then, even if the game exhibits strategic complements, the noise can vanish in the unstable direction. Consider the 2-player game with with*

$$u_1(x_1, x_2) = -\frac{x_1^2}{2} + 2x_1 - x_1(2 - x_2)^2; \;\; u_2(x_1, x_2) = -\frac{x_2^2}{2} - x_1^2(2 - x_2).$$

[22] we need to make sure that r is small enough such that $\overline{V_r} = P^{-1}([v - r, v]) \subset U$

It is not hard to see that $(1,1)$ *is a Nash equilibrium, and that*

$$\frac{\partial^2 u_i}{\partial x_i \partial x_j}(\hat{x}) = 2, \;\; i = 1, 2.$$

As a consequence the Jacobian matrix associated to the dynamics G *is simply*

$$DG(\hat{x}) = \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix},$$

which corresponds to a saddle point, as the eigenvalues are -3 *and* 1. *The eigenspace associated to the positive eigenvalue is generated by* $v = (1,1)$. *Now it is not hard to see that the externalities are antisymmetric:* $\frac{\partial u_2}{\partial x_1}(x) = -\frac{\partial u_1}{\partial x_2}(x)$. *Thus, on the event* $\{\lim_n x_n = \hat{x}\}$, *we have*

$$\lim_{n\to+\infty} \mathbb{E}\left(< U_{n+1}, v >^2 \middle| \, \mathcal{F}_n\right) = \left(\frac{\partial u_1}{\partial x_2}(\hat{x}) + \frac{\partial u_2}{\partial x_1}(\hat{x})\right)^2 = 0$$

and the noise condition does not hold.

As mentioned earlier, by a well-known result of Benaïm (references to be added), on the event $\{\limsup_n \|x_n\| < +\infty\}$, the limit set of (x_n) is always compact, invariant and attractor-free. This class of sets is called *internally chain transitive* (ICT) (references to be added). Zeroes of G, periodic orbits and more generally the omega limit set of any point x (if not empty) are ICT sets. However some more complicated sets also are ICT: for instance any connected set of zeroes of G. In order to be able to claim that $(x_n)_n$ converges to a Nash equilibrium, we need additional information on the game.

For games with complements, because of the properties of the payoff functions, our dynamics enjoys nice properties and we indeed know more about the internally chain transitive sets. The first part of the next Theorem is proved in Benaïm (2000) (see the proof of Proposition 3.2) and the second part restates Theorem 3.3 in the same paper (relying heavily on Hirsch (1999)) (references to be added). We state the general result for a real analytic function F on $\mathbb{R}^d$.

Theorem 8 *Let* $K \subset \mathbb{R}^d$ *be a compact invariant set and suppose that* F *is analytic, cooperative and irreducible in* K. *Then any internally chain transitive set in* K *is either a single equilibrium* $x^* \in Z^S(F)$ *or is contained in a repulsive normally hyperbolic set.*

For sake of simplicity, we do not give the exact definition of a repulsive normally hyperbolic set. Intuitively these sets generalize the concept of linearly unstable equilibrium. These sets are contained in invariant manifolds (i.e. the stable manifold in the case of linearly unstable equilibria).As already mentioned above, strategic complements imply that our dynamics $\dot{x} = G(x)$ is cooperative. This is also true for the best response dynamics, by a direct application of the implicit function theorem. We have more:

Lemma 5 *Assume that the payoff functions are real analytic and satisfy Hypothesis 2 and 3. Then G and Br are analytic. Also the best response dynamics $\dot{x} = -x + br(x)$ and the gradient dynamics $\dot{x} = G(x)$ are cooperative and irreducible.*

Proof. We only need to prove that $\dot{x} = -x + Br(x)$ is cooperative and irreducible, as this obviously holds for $\dot{x} = G(x)$. Let $F(x) = -x + Br(x)$. According to the implicit function theorem, and since $Br_i(x_{-i}) > 0$,

$$\frac{\partial BR_i}{\partial x_j}(x_{-i}) = -\frac{\frac{\partial^2 u_i}{\partial x_i \partial x_j}(BR_i(x_{-i}), x_{-i})}{\frac{\partial^2 u_i}{\partial x_i^2}(BR_i(x_{-i}), x_{-i})} \geq 0,$$

and irreducibility follows from Hypothesis 3. ■

Finally since the graph is non-bipartite, by a simple adaptation of the proof of Theorem 3, the noise is uniform. Then Theorem 7 follows from a direct application of Theorem 4.4 and Corollary 4.6 in Benaïm and Faure (2012).

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