EXIT GAME WITH INFORMATION EXTERNALITIES

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Abstract. I analyze a two-player stopping time game with pure informational externalities. While the players are in the game, they receive deterministic revenues and incur stochastic costs. The cost process is common to both players. Each player incurs the cost at random times. Times of arrival of costs are modeled as Poisson processes with player-specific parameters. The Poisson processes are independent of each other and of the cost process. The Poisson processes represent idiosyncratic parts of risks. Each player learns about the current value of the cost both when she incurs the cost, and when the other player incurs the cost. Thus, each player benefits if the other player stays in the game longer, because this increases the frequency of observations and the value of staying in the game. As a result, players remain active longer than a single player would do. I demonstrate that if the players are heterogeneous, there is an equilibrium, where they exit the game sequentially, and the order of exit is determined endogenously. I show that if at the time of the news arrival it is not optimal to act, then it is optimal for player i to fix a time $T_i(x)$ that depends on the observed realization of the shock x, and exit at time $T_i(x)$ unless the new piece of information arrives at $\tau \le T_1(x) \wedge T_2(x)$. This is a qualitatively new result both in the strategic learning/experimentation and optimal stopping literature.

Keywords: optimal stopping, jump-diffusion process, Poisson arrival, strategic exit

JEL: C73, C61, D81

1. Introduction

1.1. **Objectives and the main result.** Timing is a key feature of many economic decisions such as financing an innovation, marketing a new product, or abandoning a piece of equipment prone to breakdowns. There are two major strands of literature that deal with timing of actions in a risky environment. The real options literature emphasizes the fact that performing an irreversible action (for example, exit from a declining industry) when payoffs are stochastic involves sacrificing the option to

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perform this action in the future. The optimal timing rule is formulated in terms of a stopping time, i.e., the first time the underlying stochastic variable enters a certain state space region, called action or stopping region. In such models, information either arrives continuously or at equally distanced time moments. However, in reality, news about such events as new drugs discoveries, stock market crashes, terrorism acts, or environmental catastrophes arrive at random times. Therefore, the first question I ask in the paper is: "given that observations arrive at random times, is it always optimal to undertake an irreversible action only when one observes a realization of the state?" If the answer to the first question is "no," then when is it optimal to act?

The insights of the literature on experimentation and learning indicate that if the information about, say, quality of a new technology, arrives at random times, it may be optimal to take an action such as adoption of the risky technology between two consecutive observations. In models based on two-armed Poisson bandits, it is typically assumed that the "safe" arm of a bandit generates a known payoff, and the "risky" arm generates an unknown payoff. This unknown payoff may, for example, characterize the quality of the new technology or a venture capital (VC) project. The new technology may be "good" or "bad" in terms of frequency of breakdowns. Similarly, the VC project may be "good" or "bad" in terms of frequency of breakthroughs. Thus, in these models, the quality of a project, equipment, or technology is intrinsically "good" or "bad," and an experimenter tries to assess the failure or success rates of a project. While waiting for an observation to arrive, the DM updates her beliefs about the rate of success or failure and takes an action when the posterior belief reaches a certain cut-off. In particular, if the "good" technology never breaks down, the first breakdown will induce the experimenter to abandon the risky technology.

The settings of the above models ignore such features as dynamics of costs of breakdowns or profits of breakthroughs. For example, nuclear stations may have easily repairable breakdowns, or breakdowns on the scale of Chernobyl. When making decisions about optimal abandonment of a project with costly breakdowns, one takes into account not only the frequency of failures, but also the cost incurred after each breakdown, and whether the manufacturer successfully (say, a number of cars recalled decreases) or unsuccessfully (say, a new version of software fails no matter what) improves the equipment. Similarly, when a breakthrough happens, the profitability of the project may depend on specific market conditions (e.g., new shale oil extraction technology may be more or less profitable depending on oil prices). I believe that in order to account for such features, instead of considering two-armed bandits who are either "white knights" or "black villains," it is necessary to consider bandits of "evolving shade of grey." To this end, I introduce a model with random costly breakdowns, where the costs of breakdowns arrive at random times that follow the Poisson process, and the cost itself follows a stochastic process independent of the Poisson process. In this setting, I consider a two player exit game with informational externalities I ask the following questions: (i) If a breakdown or breakthrough is conclusive, is it optimal to act immediately at the time of the first observation? (ii) If not, when is it optimal to act? (iii) When is it optimal to act in case of inconclusive failures or successes? (iv) How information externalities and strategic interactions influence actions timing?

I show that if at the time of the news arrival it is not optimal to act, then it is optimal for player i to fix a time $T_i(x)$ that depends on the observed realization of the shock x, and exit at time $T_i(x)$ unless the new piece of information arrives at $\tau \leq T_1(x) \wedge T_2(x)$. Moreover, even if the first breakdown is conclusive, immediate action is not always optimal after the first failure. This is a qualitatively new result both in the strategic learning/experimentation and optimal stopping literature.

In models with strategic learning, players impose information externalities on their counterparts as long as they keep experimenting. In my model, there are two sorts of information externalities: when one of the firms incurs a breakdown, both firms not only update their posterior beliefs about the rate of arrival of failures, but also they learn about the state of the underlying cost process. That is why it may be optimal to stay in the business even in the case of conclusive breakdowns after the first failure is observed.

1.2. **Empirical facts.** Here are some examples that demonstrate that not only the occurrence of a costly event, but also the size of the loss matters, and actions not necessarily take place right after an observation of a random cost.

1. Before the 9/11 terrorist attacks, the terrorism risk was included as an unnamed peril into commercial insurance contracts in the U.S. Such events as the first bombing of the World Trade Center in New York in 1993, or the 1998 bomb attacks on the U.S. embassy in Nairobi, Kenya, which caused significant insurance losses (see [34] for details) did not change the attitude of the U.S. insurers or international reinsurers to terrorism coverage. The situation changed dramatically after 9/11 attacks, which resulted in unprecedented losses. Private reinsurers, who covered the majority of these losses exited the market, and a few months after the attacks, the insurers excluded terrorism from their policies in most states (see [34] for details).

2. Other examples of how insurers who suffered large losses from a disaster are reluctant to continue offering coverage against this risk are hurricane insurance market in Florida and earthquake insurance market in California. Even though hurricanes in Florida are not rare events, the especially large losses during 2004 and 2005 hurricane seasons caused a failure of private insurance market. In California, private insurance companies decided to stop covering the residential property after the Northridge earthquake of 1994. Moreover, the earthquake happened in January 1994, but private insurance companies decided to quit the residential property market only in 1995. Hence, a one year delay was observed.

3. Following the shortage of reinsurance after such catastrophic events as Hurricane Andrew in 1992 and Northridge earthquake in 1994, there emerged a special financial contracts in order to complement reinsurance in covering large losses. One of such contracts are catastrophe (CAT) bonds. All CATs are structured to pay on triggers; in particular, there are CAT bonds that pay on insurer-specific catastrophe losses, or insurance-industry catastrophe loss indices (see, e.g., [21] for details).

4. Before the Fukushima 2011 disaster, 442 nuclear power reactors in 30 countries produces 14 per cent of all world's electricity. This number dropped to 11 per cent in 2012 as 15 reactors, mainly in Germany and Japan, exited service.

1.3. **Literature review.** The insights of the real options literature have significantly influenced capital budgeting decisions of corporations in the recent decades. In the majority of models of optimal exercise of American options and real options (real investment problems), it is presumed that the underlying process is observed either continuously (see, e.g., [22, 10, 16, 17]) or at equally distanced discrete time moments, when nothing can happen between any two consecutive observations (see, e.g., [15, 14, 17]). In reality, in many instances, the information flow is neither continuous, nor discrete. In fact, financing decisions are affected not only by uncertainty of the underlying payoff process, but also by uncertainty about the time of arrival of the next piece of news. Such news may be, for example, a breakthrough innovation, an evidence of efficiency of a new drug, a discovery of a new gas or oil site. All such pieces of information arrive at random times and they may radically change a financing decision of a related project. The adoption of new technologies crucially hinges upon an assessment of the risks they might entail. In this case, the critical events that convey new information are breakdowns of new equipment or technology which also happen at random times and also influence decisions about which technology to use.

The simplest way to model the uncertainty in the time of arrival of the next piece of news is to use a Poisson process, and assume that the option exercise (investment) is possible only at times of arrival of a piece of news. By now, there is a number of papers, which solve the corresponding problem in several situations such as portfolio optimization problem and option exercise [26, 28, 20, 42]. However, in these papers, no justification of optimality of actions restricted to the times of observations is provided.

At the same time, there is extensive literature on learning and experimentation that uses the multi-armed bandit framework. Bandit problems are used to study the trade-off between exploration and exploitation. Bandit models were successfully used in various settings in economics, for example, learning and matching in labor markets, monopolist pricing with unknown demand, choice between R&D projects, or financing of innovations (see, e.g., [3, 4, 5, 6, 7, 8, 31, 44, 45, 47] and references therein). The situation becomes even more interesting if several DM's participate in experimentation because in this case information externalities are present, and, potentially, free riding problems may arise. See, for example, [9, 23, 29, 30, 32, 33].

In models based on two-armed bandits, the decision maker (DM) has to decide on optimal allocation of her time between the safe action (arm of the bandit) and the risky action (arm). The safe action generates a payoff given by a known distribution (in many instances, this payoff is deterministic), and the risky action generates an unknown payoff. In continuous time models, the latter payoff follows a certain continuous time stochastic process whose parameters are not known. For example, Bolton and Harris [9] model the unknown payoff as a Brownian motion with unknown drift and known variance in a model of strategic experimentation. Decamps et al. [24, 25] study timing a fixed size investment into a risky project with the payoff generated by a Brownian motion with unknown drift and known variance. Keller et al. [30], Keller and Rady [32, 33] use a Poisson process with unknown rate of arrival to model the risky arm. Decamps and Mariotti [23] study a duopoly model of investment where a signal about the quality of the project is modeled as a Poisson process. Cohen and Solan [19] bridge the gap between the Brownian motion and Poisson bandits and consider two-armed bandits, where the risky arm yields stochastic payoffs generated by a Lévy process. In two-armed bandit problems, it is typically assumed that the expected payoff generated by the risky arm is higher (respectively, lower) than the expected payoff generated by the safe arm, if the risky arm is, in some sense, "good" (respectively, "bad"). The optimal stopping rule in such problems is of a cut-off type: as long as the posterior belief that the payoff distribution of the risky arm is "good" has not reached the cut-off, the DM continues experimentation with the risky arm. When the posterior belief reaches the cut-off, the DM switches to the safe arm. In particular, in learning models with conclusive breakthroughs or breakdowns, this means that the first observation of a success or a failure implies immediate action (for example, adoption or discarding of a risky technology).

1.4. **The structure of the paper.** The rest of the paper is organized as follows. In Section 2, I set up the model of strategic exit and as a benchmark, Iconsider exit in a duopoly in a model with conclusive breakdowns after the first costly breakdown had taken place. In this case, the players have learned the rate of arrival of breakdowns and only have to decide on timing exit. I assume that the firms will get the same recovery values at the moment of exit. When active, the firms are heterogeneous in terms of the expected present values (EPV) of the profits. Namely, after any observation of the costs incurred by one of the players, one of the firms has smaller EPV of the profits than the other one. This guarantees that the firm with the smaller EPV will be the first one to exit (or the leader).

In Section 3, I start with the case of a single decision-maker who determines the optimal timing of abandoning a project with costly breakdowns. Integral representation of the value function of a non-strategic DM is recursive. I present a convenient recursive procedure which calculates simultaneously the optimal stopping strategy and value function. I need this case to find the leader's exit strategy. Since the roles of the players are predetermined as specified above, the leader's exit decision is non-strategic. At the same time, before exiting, the future leader enjoys the information externality, because the observations of the cost process arrive more frequently, when two players are active. I provide sufficient conditions under which exit is not optimal at the moment of observations. Instead, player i finds an optimal time $T_i(x)$ that depends on the observed realization of the shock x, and exits at time $T_i(x)$ unless the new piece of information arrives at $\tau \le T_1(x) \wedge T_2(x)$.

Notice that if the leader exits at some time between two consecutive observations, the follower will have to formulate the exit strategy when the state of the cost process is unobserved. In this case, the integral representation of the value function is no longer recursive, but it involves averaging of the value functions with observable initial state. This case is studied in Section 3.3.

In the future work, I will consider the case when the players have to update their beliefs about the rates of arrival of breakdowns.

Technical details are relegated to the Appendix.

2. Model environment and preliminary results

2.1. **Setting.** Two players (insurance companies) operate in areas potentially subject to costly disasters (for example, flooding). Each area may be "good" or "bad" depending on the frequency of disasters. Costs incurred by company j arrive at random times that follow the Poisson process N^j with rate of arrival λ_j. In this paper, I assume that Poisson processes N_1 and N_2 are independent of each other. The reader may think about one insurance company operating on the New Jersey coast, and the other - on the Gulf coast. Costs of disasters are publicly observed. Rates of arrivals of disasters are initially unknown to either player and can take one of two possible values: $\lambda''_j > \lambda'_j \geq 0$. Hence, the area of operation of company j is "bad" (respectively, "good") if λ''_j (respectively, λ'_j) is the true rate of arrival. If $\lambda'_j = 0$, then after the first disaster hits player j, the region of operation of j is conclusively bad. The initial common prior assigns probability $\bar{p}_j$ to λ'_j.

Time is continuous. First, we consider the evolution of beliefs $(p_1(t), p_2(t))$ until the next disaster strikes, at a random time τ. Assuming $(p_1(t), p_2(t))$ are known, and the disaster does not happen during the next infinitesimally small time interval Δt, updated beliefs are

$$p_j(t + \Delta t) \quad = \quad \text{Prob}\,(\lambda_j = \lambda'_j \mid \tau > t + \Delta t) = \frac{p_j(t)e^{-\lambda'_j \Delta t}}{p_j(t)e^{-\lambda'_j \Delta t} + (1 - p_j(t))e^{-\lambda''_j \Delta t}} + o(\Delta t).$$

Calculating the derivative of the RHS w.r.t. Δt at $\Delta t = 0$ we derive the ordinary differential equation for the evolution of beliefs: $(p_j)'(t) = -p_j(t)(1 - p_j(t))(\lambda'_j - \lambda''_j)$.

The solution is standard and well-known: if $p_j(0) = \bar{p}_j$, then, until the moment of the first observation, p_j and λ_j evolve according to

$$p_j(\bar{p}_j; t) \quad = \quad \frac{\bar{p}_j}{(1 - \bar{p}_j) \cdot e^{t(\lambda'_j - \lambda''_j)} + \bar{p}_j}, \tag{2.1}$$

$$\lambda_j(\bar{p}_j; t) \quad = \quad \lambda''_j + \frac{\bar{p}_j(\lambda'_j - \lambda''_j)}{(1 - \bar{p}_j) \cdot e^{t(\lambda'_j - \lambda''_j)} + \bar{p}_j}. \tag{2.2}$$

At the next moment of observation τ, p_j is updated as follows. For $0 \leq t < \tau_j$, $p_j(\bar{p}_j; t)$ is known - see (2.1), and the limit $p_j(\bar{p}_j; \tau-)$ is well-defined. Using the Bayes rule, we obtain the updated belief at the moment of the next observation.

$$p_{j,u}(\bar{p}_j; \tau) = \lim_{\Delta t \downarrow 0} \frac{p_j(\bar{p}_j; \tau - \Delta t)(1 - e^{-\lambda'_j \Delta t})}{p_j(\bar{p}_j; \tau - \Delta t)(1 - e^{-\lambda'_j \Delta t}) + (1 - p_j(\bar{p}_j; \tau - \Delta t))(1 - e^{-\lambda''_j \Delta t})}.$$

The result is

$$p_{j,u}(\bar{p}_j, \tau) = \frac{p_j(\bar{p}_j, \tau)\lambda'_j}{p_j(\bar{p}_j; \tau)(\lambda'_j - \lambda''_j) + \lambda''_j}, \tag{2.3}$$

where $p_j(\bar{p}_j; \tau)$ is given by (2.1). After that, the posterior $p_{j,u}(\bar{p}_j, \tau)$ is substituted instead of $\bar{p}_j$ into (2.1) and (2.2), which gives updated functions $p_j(\bar{p}_j; t)$ and $\lambda_j(\bar{p}_j; t)$ for the next period between observations. Note that beliefs $(p_j(\bar{p}_j; t), 1 - p_j(\bar{p}_j; t))$ define a time-dependent probability measure on the state space $(\lambda'_j, \lambda''_j)$. If $\lambda'_j = 0$, then after the first observation, $p_{j,u}(\bar{p}_j, \tau) = 0$, and $\lambda_j(\bar{p}_j, \tau) = \lambda''_j$.

For simplicity, assume that an active insurance company j gets the stream of instantaneous deterministic profits $R_j > 0$. The riskless rate is $r > 0$. I model the instantaneous cost incurred by firm $j \in \{1, 2\}$ at time t as $A_j e^{X_t}$, where $A_j > 0$, and $X = \{X_t\}_{t \geq 0}$ is a jump-diffusion process with i.i.d. increments (Lévy process) on $\mathbb{R}$. On the filtered measurable space generated by X, we choose a probability measure $\mathbb{Q}$ so that the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq \infty}, \mathbb{Q})$ satisfies the usual properties.[1] See [46] for the definitions and Section A.1 for main facts about Lévy processes. If $\mathbb{E}^{\mathbb{Q}}\left[e^{X_1}\right] < \infty$, then the Lévy exponent $\Psi := \Psi^{\mathbb{Q}}$ of X under $\mathbb{Q}$ is definable from $\mathbb{E}^{\mathbb{Q}}\left[e^{\beta X_t}\right] = e^{t\Psi(\beta)}$ for $\beta \in [0, 1]$. Assume that the supremum process $\bar{X}_t = \sup_{0 \leq s \leq t} X_s$ is non-trivial. Assume that the cost process X is independent of the Poisson processes N^1 and N^2. Between the jumps in N^1 or N^2, no new piece of information arrives. I assume that the disaster never strikes both firms simultaneously (in the future work, I plan to analyze how this assumption can be relaxed).

The time of the last observation is normalized to 0. Let $\tau_j \sim \operatorname{Exp} \lambda_j$ be the random time of arrival of the next disaster for firm j. Let $x = X_0$ denote the value of X at the moment of the last observation.

Lemma 2.1. *The expected present value $C_j(x)$ of the cost is finite if and only if $\Psi(1) < +\infty$ and*

$$r + \lambda_j - \Psi(1) > 0. \tag{2.4}$$

If (2.4) holds, $C_j(x) = \lambda_j A_j e^x / (r + \lambda_j - \Psi(1))$.

Proof. We have

$$\begin{aligned} \mathbb{E}^{\mathbb{Q},x}\left[e^{-r\tau X_\tau}\right] &\equiv \mathbb{E}^{\mathbb{Q}}\left[E^{-r\tau X_\tau} \,|\, X_0 = x\right] \\ &= \int_0^{+\infty} \lambda_j A_j e^{-(r+\lambda_j)t} \mathbb{E}^{\mathbb{Q},x}\left[e^{X_t}\right] dt = \int_0^{+\infty} \lambda_j A_j e^{-(r+\lambda_j)t} e^{t\Psi(1)+x} \, dt. \end{aligned}$$

The integral converges if and only if (2.4) holds[2]. □

In the case of exit at time t_j, company j gets value $G_j(X_t)$. For simplicity assume that the exit value is constant, and it is a fraction of the value of the perpetual stream R_j. Thus, $G_j(X_t) = \alpha_j R_j / r$, where $\alpha_j \in [0, 1)$, $j \in \{1, 2\}$. Although the nature of

[1] Namely, $\mathcal{F}_0$ contains all the P-null sets of $\mathcal{F}$, and the filtration $(\mathcal{F}_t)_{0 \leq t < \infty}$ is right continuous.

[2] The more rigorous notation would be $\mathbb{E}^{\mathbb{Q} \otimes \mathbb{P}, x}$ instead of $\mathbb{E}^{\mathbb{Q},x}$, where $\mathbb{P}$ is the probability measure associated with τ. We use $\mathbb{E}^{\mathbb{Q},x}$ to simplify the notation.

shocks is the same, hence, the process for the size of an individual disaster is the same, the locations where the firms operate are different, hence, the hazard rates, instantaneous profits, costs and recovery rates are different: λ_j, R_j, $C_j(X_t) = A_j e^{X_t}$ and $\alpha_j, j = 1, 2$.

One can also consider the costs $C_j(X_t)$ which are decreasing functions of the shock, and which have sufficiently large limits at infinity. The techniques of the paper is applicable to this more general situation as well. The Bellman equations for the exit thresholds will involve functions $\mathcal{C}_j(T, x) = \mathbb{E}^x[C_j(X_T)]$. The condition that I need is that $\mathcal{C}_j(T, x)$ are increasing in both arguments, and $\mathcal{C}_j(T, x)$ tends to a sufficient large limit (non-necessarily infinite) as $T \to +\infty$.

Remark 2.2. This setting presumes that the company can always bear the cost, which is not quite realistic. However, the methodology of the paper is applicable to more realistic situations. It is possible to model the size of a disaster as a function of X that is bounded from above or increases very slow. Then, assuming that the company has sufficiently deep pockets, the question of the default does not arise; to be more accurate, the probability of default can be disregarded as being too low.

The cost of a disaster for the insurance company j is bounded if the cost above a certain level H_j is covered by reinsurers. In such setting, it would be necessary to assume that R_j is the profit flow net of reinsurance costs and that, after each disaster, the cost of reinsurance jumps up (hence, R_j jumps down), and, possibly, H_j jumps up. The natural assumption would be that, when a disaster of a too large scale happens, the reinsurer will abandon the client, and the company will remain on its own. In this case, it would be natural to introduce the default level for the insurance company. One can start with the company that does not use a reinsurer but defaults when a certain level H_j for the costs is reached. Notice, however, that if the insurance company is large, and can withstand significant costs, such complications (which are possible to analyze) do not lead to a sizable change in the optimal exit rule. Therefore, the qualitative conclusions of this section and the following ones will not change.

I will consider a stopping time game between the two players, where as long as both companies are active, each player has only two available strategies - stop (or exit) a continue. If player i stops at time t_i and player j stops later, then player i is the leader, and player j is the follower. In the stopping game with randomly incurred stochastic costs, the exit strategy will depend on the last observation of the cost process X. Let $U_j = \{x \mid T_j(x) > 0\}$ be the waiting (or continuation) region of the company j at the time of the last observation normalized to 0. Then $U_j^c = \mathbb{R} \setminus U_j$ is the stopping region of the company j at the moment of the last observation.

The exit strategy of player j is specified by a function $T_j : \mathbb{R} \to [0, +\infty)$ as follows.

(1) If $T_j(x) = 0$, equivalently, if $x \in U_j^c$, exit immediately.
(2) If $T_j(x) > 0$, and the next moment of observation $\tau > T_j(x)$ (the disaster did not happen by time $T_j(x)$), exit at time $T_j(x)$.

(3) If $T_j(x) > 0$, and the next moment of observation $\tau \leq T_j(x)$, start the game against the nature anew, with X_τ as the new spot level of the shock at time 0.

I look for a Markov perfect equilibrium (MPE) as a profile of exit strategies which are mutually best responses.

2.2. **Sequential exit.** In this Section, I assume $\lambda_1' = \lambda_2' = 0$, so for both companies the disasters are conclusive. Further, assume that both companies had been hit by at least one disaster, so they know the true rates of arrival $\lambda_1 > 0$ and $\lambda_2 > 0$. To simplify the analysis of the exit strategies, let the recovery values of the companies be the same: $\alpha_1 R_1 = \alpha_2 R_2$. Finally, assume that the expected present value (EPV) of player 1's profit is less than the EPV of player 2's profit for any last observation x and any $t > 0$ s.t. both players are active until t:

$$\begin{aligned}\Pi_1(x,t) &= \mathbb{E}^x\left[\int_0^t e^{-rs}\left(R_1 - \lambda_1 A_1 e^{X_s}\right)ds\right] \\ &< \mathbb{E}^x\left[\int_0^t e^{-rs}\left(R_2 - \lambda_2 A_2 e^{X_s}\right)ds\right] = \Pi_2(x,t).\end{aligned}$$

Let $V_j(x)$ denote the strategic value of player j at the moment of last observation, and τ^{12} denote the random time when a disaster hits at least one player.

Lemma 2.3. *Let* $\Pi_1(x,t) \leq \Pi_2(x,t)\ \forall\,(x,t)$ *and* $T_1(x) > 0$. *Then* $T_1(x) \leq T_2(x)\ \forall\, x$.

Proof. Consider $0 < t < T_1(x) \wedge \tau^{12}$. Then

$$\mathbb{E}^x\left[e^{-rt}V_1(X_t)\right] > \frac{\alpha_1 R_1}{r} = \frac{\alpha_2 R_2}{r}.$$

Since $\Pi_1(x,t) \leq \Pi_2(x,t)$, $\mathbb{E}^x\left[e^{-rt}V_2(X_t)\right] \geq \mathbb{E}^x\left[e^{-rt}V_1(X_t)\right]$, hence exit at t cannot be optimal for player 2, hence $T_1(x) \leq T_2(x)$. □

Since the roles of the players are predetermined by the assumptions of the model, we see that both players will make non-strategic exit decisions. The difference between the leader and the follower is that the leader gets information about the cost process with frequency $\lambda_1 + \lambda_2$ and makes the exit decision after at least one observation of the process X. If the leader exits at some time between two consecutive observations, the follower will make the non-strategic exit decision with unknown state x, and after the leader's exit, the frequency of observations will be reduced to λ_2.

3. Conclusive disasters, non-strategic exit

3.1. **Known initial state.** The first building block for the strategic exit problem is to consider a non-strategic exit decision after the first conclusive disaster had been observed. For the time being, I will suppress the subscript j that indicates the player. I start with the following

Lemma 3.1. *The waiting region* U *is of the form* $(-\infty, \bar{h})$. *The value function* V *is decreasing on* U.

Proof. Let $x_2 \in U$ and $x_1 < x_2$. Let $T(x_2) > 0$ be an optimal exit strategy given the last observation x_2. Suppose that when the last observation is x_1, the company decides to exit at $T(x_2)$, which may be suboptimal. With a positive probability the next cost will be incurred before $T(x_2)$, and the costs incurred will be

$$e^{x_1+X_N(\omega)} < e^{x_2+X_N(\omega)}, \quad \forall \omega \in \Omega^N,$$

where the process X_N starts at 0. Hence, the value function at x_2 with the suboptimal exit rule $T(x_2)$ is greater than $V(x_2)$. Choosing the optimal exit strategy at x_1 cannot decrease the firm's value, therefore $V(x_1) > V(x_2) \geq \alpha R/r$, and it is non-optimal to exit at $X_0 = x_1$. □

Let $W(\mu, \lambda, V; T, t, x)$ denote the value function of the company that gets observations with rate μ, incurs costs with rate λ, suffered the last cost e^x at $t = 0$, remained active until $t > 0$ and plans to exit at $T > t$ unless a new disaster strikes (this company or the other one) at $\tau < T$. For all $t < T \wedge \tau$, the only information is the one available at time 0. Applying the standard infinitesimal accounting techniques, we calculate

$$\begin{aligned} W(\mu, \lambda, V; T, t, x) &= \int_t^T e^{-(s-t)(r+\mu)} R ds + \frac{\alpha R}{r} e^{-(T-t)(r+\mu)} \\ &\quad + \int_t^T e^{-(s-t)(r+\mu)} \mu \mathbb{E}^{\mathbb{Q},x} \left[V(X_s) - \frac{\lambda}{\mu} A e^{X_s} \right] ds, \end{aligned} \tag{3.1}$$

where the first two terms on the RHS are, respectively, the EPV of the stream of revenues the company receives until the exit at time T and the recovery value at time T if $T < \tau$; the last term is the EPV of the new value net of the costs at the moment of the next disaster if the disaster happens at $\tau < T$. Simplifying, we get

$$\begin{aligned} &W(\mu, \lambda, V; T, t, x) = \frac{R}{r+\mu} \left(1 - e^{-(T-t)(r+\mu)}\right) + \frac{\alpha R}{r} e^{-(T-t)(r+\mu)} \\ &+\mu \int_t^T e^{-(s-t)(r+\mu)} \mathbb{E}^{\mathbb{Q},x} \left[V(X_s)\right] ds - \frac{\lambda e^{x+t\Psi(1)}}{r+\mu-\Psi(1)} \left(1 - e^{-(T-t)(r+\mu-\Psi(1))}\right). \end{aligned}$$

Naturally, for any x, $W(\mu, \lambda, V; 0, 0, x) = \alpha R/r$, $V(x) \geq \alpha R/r$, and

$$V(x) = \sup_{T \geq 0} W(\mu, \lambda, V; T, 0, x).$$

Let $\widetilde{V}(x) = V(x) - \alpha R/r$ be the option value of being active. Define $\widetilde{W}(\mu, \lambda, \widetilde{V}; T, t, x) = W(\mu, \lambda, \widetilde{V} + \alpha R/r; T, t, x) - \alpha R/r$. Straightforward calculations show that

$$\begin{aligned} &\widetilde{W}(\mu, \lambda, \widetilde{V}; T, t, x) = \frac{(1-\alpha)R}{r+\mu} \left(1 - e^{-(T-t)(r+\mu)}\right) \\ &+\mu \int_t^T e^{-(s-t)(r+\mu)} \mathbb{E}^{\mathbb{Q},x} \left[\widetilde{V}(X_s)\right] ds - \frac{\lambda e^{x+t\Psi(1)}}{r+\mu-\Psi(1)} \left(1 - e^{-(T-t)(r+\mu-\Psi(1))}\right). \end{aligned} \tag{3.2}$$

Furthermore, for any x, $\widetilde{W}(\mu,\lambda,\widetilde{V};0,0,x)=0$, $\widetilde{V}(x)\ge 0$, and

$$\widetilde{V}(x)=\sup_{T\ge 0}\widetilde{W}(\mu,\lambda,V;T,0,x). \tag{3.3}$$

Set $\widetilde{V}_0(x)=0$, and define, inductively, for $n=0,1,2,\ldots$,

$$\widetilde{V}_{n+1}(x)=\sup_{T\ge 0}\widetilde{W}(\mu,\lambda,\widetilde{V}_n;T,0,x),\quad x\in\mathbb{R}. \tag{3.4}$$

Theorem 3.2. *The following statements hold:*

(i) The sequence $\{\widetilde{V}_n\}_{n\in\mathbb{N}}$ is non-decreasing (point-wise).

(ii) The sequence $\{\widetilde{V}_n\}_{n\in\mathbb{N}}$ is uniformly bounded from above by $(1-\alpha)R/r$.

(iii) For any $x\in\mathbb{R}$, the limit $\widetilde{V}(x)=\lim_{n\to\infty}\widetilde{V}_n(x)$ exists, $\widetilde{V}$ admits the bound $0\le\widetilde{V}(x)\le(1-\alpha)R/r$, (3.3) holds, and $V=\widetilde{V}+\alpha R/r$ is the value function.

Proof. (i) is evident from (3.2). (ii) It suffices to prove that $\widetilde{W}(\mu,\lambda,\widetilde{V};T,t,x)$ is bounded by $(1-\alpha)R/r$ if $\widetilde{V}$ on the RHS of (3.2) is bounded by $(1-\alpha)R/r$. Replacing $\widetilde{V}$ with $(1-\alpha)R/r$ and omitting the last negative term in (3.2), we obtain

$$\begin{aligned}\widetilde{W}(\mu,\lambda,\widetilde{V};T,t,x) &\le \frac{(1-\alpha)R}{r+\lambda}\left(1-e^{-(T-t)(r+\lambda)}\right)+\frac{(1-\alpha)R}{r}\int_t^T e^{-(s-t)(r+\lambda)}ds\\ &= \frac{(1-\alpha)R}{r+\lambda}\left(1-e^{-(T-t)(r+\lambda)}\right)+\frac{(1-\alpha)R}{r(r+\lambda)}\left(1-e^{-(T-t)(r+\lambda)}\right)\\ &= \frac{(1-\alpha)R}{r}\left(1-e^{-(T-t)(r+\lambda)}\right)\le\frac{(1-\alpha)R}{r}.\end{aligned}$$

(iii) The existence of the limit and bounds on $\widetilde{V}$ follow from (i) and (ii). Passing to the limit in (3.4), we obtain (3.3). Hence, $V=\widetilde{V}+\alpha R/r$ is the value function. □

Clearly, if $T(x)=T(\mu,\lambda;x)$ maximizes $\widetilde{W}(\mu,\lambda,\widetilde{V};T,t,x)$, it also maximizes $W(\mu,\lambda,V;T,t,x)$. Denote by $T_*(x)$ (respectively, $T^*(x)$) the infimum (respectively, supremum) of optimal $T(x)$; it is possible that $T_*(x)=+\infty$ and/or $T^*(x)=+\infty$. To study the set of optimal $T=T(x)\ge 0$, we calculate the derivative $\widetilde{W}_T:=\partial\widetilde{W}/\partial T$:

$$\begin{aligned}\widetilde{W}_T(\mu,\lambda,\widetilde{V};T,t,x) &= (1-\alpha)Re^{-(T-t)(r+\mu)}+\mu e^{-(T-t)(r+\mu)}\mathbb{E}^{\mathbb{Q},x}\left[\widetilde{V}(X_T)\right]\\ &\quad -\lambda e^{x+t\Psi(1)}e^{-(T-t)(r+\mu-\Psi(1))}\\ &= \mathcal{U}(\mu,\lambda;T,x)\frac{f(T)}{f(t)},\end{aligned} \tag{3.5}$$

where $f(t)=e^{-t(r+\mu)}$, and

$$\mathcal{U}(\mu,\lambda;T,x)=(1-\alpha)R-\lambda e^{T\Psi(1)+x}+\mu\mathbb{E}^{\mathbb{Q},x}\left[\widetilde{V}(X_T)\right]. \tag{3.6}$$

Clearly, $\widetilde{W}_T(\mu,\lambda,\widetilde{V};T,t,x)>0(=0,<0)\Leftrightarrow\mathcal{U}(\mu,\lambda;T,x)>0(=0,<0)$.

Proposition 3.3. *An optimal $T = T(x)$ is time-consistent: if $T(x)$ maximizes $\widetilde{W}(\mu,\lambda,\widetilde{V};T,0;x)$, then, for any $t \le T(x)$, $T(x)$ is a maximizer of $\widetilde{W}(\mu,\lambda,\widetilde{V};T,t,x)$.*

Proof. For any $0 \le t < T$, we have

$$\begin{aligned}\widetilde{W}(\mu,\lambda,\widetilde{V};T,0,x) &= \int_0^T \mathcal{U}(\mu,\lambda;t',x)f(t')/f(0)dt' \\ &= \int_0^t \mathcal{U}(\mu,\lambda;t',x)f(t')/f(0)dt' + \frac{f(t)}{f(0)}\int_t^T \mathcal{U}(\mu,\lambda;t',x)f(t')/f(t)dt' \\ &= \widetilde{W}(\mu,\lambda,\widetilde{V};t,0,x) + \frac{f(t)}{f(0)}\widetilde{W}(\mu,\lambda,\widetilde{V};T,t,x).\end{aligned}$$

Since $\widetilde{W}(\mu,\lambda,\widetilde{V};t,0,x)$ and $f(t)/f(0)(>0)$ are independent of T, a T which maximizes $\widetilde{W}(\mu,\lambda,\widetilde{V};T,0,x)$ maximizes $\widetilde{W}(\mu,\lambda,\widetilde{V};T,t,x)$ as well. □

3.2. **General properties of an optimal exit rule.**

Proposition 3.4. *1. If $\mathcal{U}(\mu,\lambda;0,x) > 0$, then it is non-optimal to exit at time 0: $T_*(x) > 0$ (possibly, $T_*(x) = +\infty$).*
2. If $\mathcal{U}(\mu,\lambda;0,x) < 0$, then there are three possibilities
a. it is non-optimal to wait and $T_*(x) = T^*(x) = 0$;
b. it is optimal to exit either immediately or later: $T_*(x) = 0$, $T^*(x) > 0$;
c. it is non-optimal to exit at time 0: $T_*(x) > 0$ (possibly, $T_*(x) = +\infty$).

Proof. 1. If $\mathcal{U}(\mu,\lambda;0,x) > 0$, then $\widetilde{W}(\mu,\lambda,\widetilde{V};T,0,x)$ increases in T in a neighborhood of 0. 2. If $\mathcal{U}(\mu,\lambda;0,x) < 0$, then $\widetilde{W}(\mu,\lambda,\widetilde{V};T,0,x) < 0 \le \widetilde{V}(x)$ for sufficiently small positive T. Consider the following three possibilities. a. $\widetilde{W}(\mu,\lambda,\widetilde{V};T,0,x) < 0$ for all $T > 0$, the global maximum is attained at $T = 0$, and the only optimal strategy is to exit instantly. b. $\widetilde{W}(\mu,\lambda,\widetilde{V};T,0,x) \le 0$ for all $T \ge 0$, and there exists $T > 0$ such that $\widetilde{W}(\mu,\lambda,\widetilde{V};T,0,x) = 0$; then the unconstrained maximum of $\widetilde{W}$ is zero, which is the same as the value of $\widetilde{W}$ at the boundary $T = 0$. In this case, the DM is indifferent between exiting immediately ($T_*(x) = 0$) and waiting until the unconstrained maximum is reached ($T^*(x) > 0$). c. $\widetilde{W}(\mu,\lambda,\widetilde{V};T,0,x) > 0$, for some $T > 0$; then any local maximum of $\widetilde{W}$ is positive, and the value of $\widetilde{W}$ at the boundary $T = 0$ is zero. Hence it is optimal to wait. □

Proposition 3.5. *(i) If $\Psi(1) \ge 0$, there exists x_+ such that if $x \ge x_+$, then it is optimal to exit at time 0: $T^*(x) = 0$;*
(ii) if $\Psi(1) \le 0$, there exists x_- such that if $x \le x_-$, then it is non-optimal to exit before the next piece of information arrives: $T_*(x) = +\infty$.

Proof. Since $0 \le \widetilde{V}(x) < (1-\alpha)R/r$ for any x, we have

$$(1-\alpha)R - \lambda e^{T\Psi(1)+x} \le \mathcal{U}(\mu,\lambda;T,x) < (1-\alpha)R\frac{r+\mu}{r} - \lambda e^{T\Psi(1)+x},\ \forall\, x \in \mathbb{R},\, T \ge 0.$$

If $\Psi(1) \geq 0$, then the rightmost part is negative for $x > \ln{(1-\alpha)R(r+\mu)/(r\lambda))}$ and $T \geq 0$. If $\Psi(1) \leq 0$, then the leftmost part is positive for $x < \ln{((1-\alpha)R/\lambda)}$ and $T \geq 0$. □

Below, we concentrate on the case $\Psi(1) > 0$; this condition is equivalent to the statement that, on average, the next disaster is more costly than the previous one.

Theorem 3.6. *Let $\Psi(1) > 0$ and $\mathcal{U}(\mu, \lambda; 0, x) > 0$. Then it is non-optimal to exit at time 0, and it is non-optimal to wait indefinitely for the next piece of information to arrive: $0 < T_*(x) \leq T^*(x) < +\infty$.*

Proof. Since $\mathcal{U}(\mu, \lambda; 0, x) > 0$, the function $\widetilde{W}(\mu, \lambda, \widetilde{V}; T, 0, x)$ increases in T in a neighborhood of 0, hence, it is non-optimal to exit at $t = 0$. Since $\widetilde{V} \leq (1-\alpha)R/r$, and $-e^{\Psi(1)T+x} \to -\infty$ as $T \to +\infty$, function $\mathcal{U}(\mu, \lambda; T, x) \to -\infty$ as $T \to +\infty$. Therefore, all points of the global maximum are on a finite interval $[a, b] \subset (0, +\infty)$. □

Theorem 3.6 gives sufficient conditions for non-optimality of exercise at time 0 and non-optimality of waiting indefinitely for the next piece of information to arrive. However, condition $\mathcal{U}(\mu, \lambda; 0, x) > 0$ is formulated in terms of the value function that can be found only numerically using the iteration procedure in Theorem 3.2.

Suppose now that, in addition, the first instantaneous moment of the process is positive: $\Psi'(0) > 0$. Then each sample path of the process X tends to $+\infty$ with probability 1 (the proof is similar to the proof of Lemma A.1 in [18]). By Theorem 3.6, $\widetilde{V}(x) = 0$ for $x \geq x_+$. Hence, for any x, $\mathbb{E}^{\mathbb{Q},x}\left[\widetilde{V}(X_T)\right] \to 0$, and $\frac{d}{dT}\mathbb{E}^{\mathbb{Q},x}\left[\widetilde{V}(X_T)\right] \to 0$ as $T \to +\infty$. Therefore, as $T \to \infty$,

$$\mathcal{U}(\mu, \lambda; T, x) \sim (1-\alpha)R - \lambda e^{T\Psi(1)+x}, \text{ and } \mathcal{U}_T(\mu, \lambda; T, x) \sim -\Psi(1)\lambda e^{T\Psi(1)+x}$$

Thus, for very large in absolute value negative x, $\mathcal{U}(\mu, \lambda; T, x)$ has the unique zero $T(x)$ that satisfies

$$T(x) \sim \frac{\ln[(1-\alpha)R/\lambda]}{\Psi(1)} - x \quad \text{as } x \to -\infty. \tag{3.7}$$

One may expect that (typically, at least), $T(x)$ is unique not only for very large positive x (when $T(x) = 0$) and very large in absolute value negative x but for other x as well.

Below we show that if X is a subordinator, that is, X is a finite variation process with non-negative drift μ and positive jumps only, then $T(x)$ is unique for all X. Denote $\bar{h} = \sup\{x \mid T_*(x) > 0\}$. This is the boundary between the action region and inaction one, at the moment of observation.

Theorem 3.7. *Let X be a subordinator. Then*

(i) $\bar{h} = \bar{h}^0$, where $\bar{h}^0$ is given by

$$e^{\bar{h}^0} = \frac{(1-\alpha)R}{\lambda}; \tag{3.8}$$

(ii) for any $x < \bar{h}$, the equation $\mathcal{U}(\mu, \lambda; T, x) = 0$ has a unique solution $T_*(x) \in (0, +\infty)$;

(iii) $T_*(\bar{h} - 0) = 0$.

See Section A.2 for the proof.

3.3. **Unknown initial state.** The next building block is to study a non-strategic exit decision when the initial state is unknown. Let $\bar{p}$ be the initial belief that the true rate of arrival is $\lambda = \lambda'$. If $\bar{p} = 1$, then the company anticipates no costs, hence, $T_*(x, \bar{p}) = +\infty$ for any x. Assume that $\bar{p} < 1$. Notice that prior to the first observation, the updated belief $\lambda(\bar{p}, t) \to 0$ as $t \to +\infty$ if $\bar{p} < 1$. At the moment of the first observation, the updated belief is $p_u = 0$.

As the first step, we assume that the initial belief about the distribution of the shock x is atomic. Mathematically, this is equivalent to the assumption that x has been observed. At the moment of the observation, not only the realization x of the cost becomes known but $\lambda = \lambda''$ as well.

Let V denote the value function in the problem considered in Section 3.1 with $\lambda = \lambda''$. Define

$$\begin{aligned} W(V; T, t, x; \bar{p}) \quad &= \quad \frac{\alpha R}{r} e^{-\int_t^T (r + \lambda(\bar{p}; s)) ds} + \int_t^T e^{-\int_t^{t'} (r + \lambda(\bar{p}; s)) ds} \\ &\quad \times \left(R + \lambda(\bar{p}; t') \mathbb{E}^{\mathbb{Q}, x} \left[V(X_{t'}) - e^{X_{t'}}\right]\right) dt'. \end{aligned}$$

Let $\widetilde{V} = V - \alpha R/r$ and $\widetilde{W}(\widetilde{V}; T, 0, x; \bar{p}) = W(\widetilde{V} + \alpha R/r; T, 0, x; \bar{p}) - \alpha R/r$. Let $\pi(dx)$ be the belief about (probability distribution of) X_0 at time 0. Introduce

$$\widetilde{W}_d(T; \pi, \bar{p}) = \int \pi(dx) \widetilde{W}(\widetilde{V}; T, 0, x; \bar{p}).$$

Then the value function at time 0 admits representation $\mathcal{V}(\pi, \bar{p}) = \alpha R/r + \widetilde{\mathcal{V}}(\pi, \bar{p})$, where

$$\widetilde{\mathcal{V}}(\pi, \bar{p}) = \sup_{T \geq 0} \widetilde{W}_d(T; \pi, \bar{p}). \tag{3.9}$$

Rewrite the last equation as

$$\widetilde{\mathcal{V}}(\pi, \bar{p}) = \sup_{T \geq 0} \int \pi(dx) \int_0^T \frac{f(s)}{f(0)} \mathcal{U}(s; x; \bar{p}) ds = \sup_{T \geq 0} \int_0^T \frac{f(s)}{f(0)} \mathcal{U}_d(s, \pi, \bar{p}) ds, \tag{3.10}$$

where

$$\mathcal{U}_d(T; \pi, \bar{p}) = (1 - \alpha) R - \lambda(\bar{p}, T) e^{\Psi(1) T} E\pi + \lambda(\bar{p}, T) \mathbb{E}^{\mathbb{Q}} \left[\widetilde{V}_d(X_T; \pi)\right].$$

Here $E\pi = \int \pi(dx) e^x$ is average size of a loss, and $\widetilde{V}_d$ is the averaged value function

$$\widetilde{V}_d(y; \pi)) = \int \pi(dx) \widetilde{V}(x + y);$$

both can be calculated numerically. Notice that time consistency of the optimal exit rule immediately follows from the form of the RHS of (3.10).

After the number $E\pi$ and function $\mathbb{E}^{\mathbb{Q}}\left[\widetilde{V}_d(X_T;\pi)\right]$ have been calculated, finding an optimal exit strategy for each pair $(\pi,\bar{p})$ becomes the standard problem of maximization of the function $\widetilde{W}_d(\cdot;\pi,\bar{p})$ on $[0,+\infty)$. To be more specific, consider the following possibilities.

1. The equation

$$\mathcal{U}_d(T,\pi,\bar{p}) = 0 \tag{3.11}$$

   has a solution $T(\pi,\bar{p})$ on $(0,+\infty)$. Then the following cases are possible:
   a. $\widetilde{W}_d(T(\pi,\bar{p}),\pi,\bar{p}) < 0$ for any $T(\pi,\bar{p})$ that satisfies (3.11), then it is optimal to exit immediately, and $T_*(\pi,\bar{p}) = T^*(\pi,\bar{p}) = 0$.
   b. $\widetilde{W}_d(T(\pi,\bar{p}),\pi,\bar{p}) \le 0$ for any $T(\pi,\bar{p})$ that satisfies (3.11), and $\widetilde{W}_d(T(\pi,\bar{p}),\pi,\bar{p}) = 0$ for some $T(\pi,\bar{p})$ that satisfies (3.11), then there is no difference between exiting immediately or at time $T(\pi,\bar{p})$, hence $T_*(\pi,\bar{p}) = 0$, and $T^*(\pi,\bar{p}) > 0$.
   c. $\widetilde{W}_d(T(\pi,\bar{p}),\pi,\bar{p}) > 0$ for some $T(\pi,\bar{p})$ that satisfies (3.11), then it is not optimal to exit at time 0: $T_*(\pi,\bar{p}) > 0$.
2. $\mathcal{U}_d(T,\pi,\bar{p}) < 0$ for all $T \ge 0$; then it is optimal to exit immediately, and $T_*(\pi,\bar{p}) = T^*(\pi,\bar{p}) = 0$.
3. $\mathcal{U}_d(T,\pi,\bar{p}) > 0$ for all $T \ge 0$; then it is never optimal to exit before the first piece of information arrives, and $T_*(\pi,\bar{p}) = T^*(\pi,\bar{p}) = +\infty$.

Since the process X is stochastically continuous,

$$\lim_{T\downarrow 0}\mathbb{E}^{\mathbb{Q}}\left[\widetilde{V}_d(X_T;\pi)\right] = \widetilde{V}_d(0;\pi,\bar{p}),$$

therefore, a sufficient condition for 1 c. is

$$\mathcal{U}_d(0,\pi,\bar{p}) = (1-\alpha)R + \lambda(\bar{p},0)(-E\pi + \widetilde{V}_d(0;\pi,\bar{p})) > 0.$$

Note that the value of expected life-time losses

$$\lambda(\bar{p},t)e^{x+t\Psi(1)} = \frac{\lambda_2(1-\bar{p})e^{x+(\Psi(1)-\lambda'')t}}{\bar{p}+(1-\bar{p})e^{-t\lambda''}}$$

tends to infinity if and only if $\bar{p} < 1$ and $\Psi(1)-\lambda'' > 0$. Therefore, the counterpart of the condition $\Psi(1) > 0$ in Theorem 3.6 (the growth rate of the costs of breakdowns is positive) is the condition $\Psi(1)-\lambda'' > 0$ (the growth rate of the costs of breakdowns is higher than their rate of arrival). If the latter condition is satisfied, the same argument as in the proof of Theorem 3.6 shows that it is not optimal to wait indefinitely for the next piece of information to arrive: $T^*(\pi;\bar{p}) < +\infty$.

4. Strategic exit: solution

Let $V_j(x)$ (respectively, $T_j(x)$) denote the strategic value (respectively, strategic exit time) of player j, and $V^0_j(x)$ (respectively, $T^0_j(x)$) denote the non-strategic value (respectively, non-strategic exit time) of the same player. By Lemma 2.3, player 2 does not exit before player 1, hence

$$\begin{aligned} V_1(x) &= V_1^0(x) \\ &= \sup_{T\geq 0} W_1(\lambda_1+\lambda_2, \lambda_1, V_1^0; T, 0, x); \\ T_1(x) &= T_1^0(\lambda_1+\lambda_2, \lambda_1; x) \end{aligned}$$

Outcome of Lemma 2.3, $T_1(x) \leq T_2(x)$, implies that (i) $T_2(x) = T_1(x) + T_2^0(\pi_{X_{T_1}|x})$

(ii) $T_1(x)$ is the termination date when the payoff $\mathcal{V}_2^0(\pi_{X_{T_1}|x})$ is due, where $\pi_{X_{T_1}|x}$ is the belief about distribution of X_{T_1} given that the last observation was x.

(iii) Since the probability of an event $\tau^{12} = T_1(x)$ is zero, it can be disregarded in all calculations.

Now we can write the value of player 2 as

$$\begin{aligned} V_2(x) = &\int_0^{T_1(x)} e^{-(r+\lambda_1+\lambda_2)t}\left(R_2 - \lambda_2 A_2 \mathbb{E}^x\left[e^{X_t}\right]\right)dt \\ &+(\lambda_1+\lambda_2)\int_0^{T_1(x)} e^{-(r+\lambda_1+\lambda_2)t}\mathbb{E}^x\left[\mathbb{1}_{X_t\notin U_1}V_2^0(X_t) + \mathbb{1}_{X_t\in U_1}V_2(X_t)\right]dt \\ &+e^{-(r+\lambda_1+\lambda_2)T_1(x)}\mathcal{V}_2^0(\pi_{X_{T_1}|x}) \end{aligned} \tag{4.1}$$

Here, no maximization is involved.

Since the RHS in (4.1) increases as V_2 increases, existence of the solution is proved by iteration method exactly as in the case of one player.

The proof also provides a numerical method: start with $V_{2,0}(x) = \alpha_2 R_2/r$, and define, inductively, for $n = 0, 1, 2, \ldots$, $V_{2,n+1}$ by the RHS of (4.1) with $V_{2,n}$ instead of V_2.

Since $V_{2,n+1} \geq V_{2,n}$ and $V_{2,n}$ are uniformly bounded by R_2/r, the limit exists.

5. Conclusion and future work

- I considered a model with costly randomly incurred disasters and demonstrated that exit at a moment of observation may be suboptimal even when disasters are conclusive.
- I derived optimal exit timing in a game with 2 heterogeneous players in case of conclusive disasters
- Future work:
  - strategic exit prior to any observation in the model with conclusive disasters;
  - exit game with inconclusive disasters;
  - games with information and payoff externalities;
  - entry game with breakthroughs.

Appendix A.

A.1. **Lévy processes and the Wiener-Hopf factorization.** We need several basic definitions and facts from [46, 17]. Recall that a Lévy process X on $\mathbb{R}$ is defined in terms of the generating triplet $(\sigma^2, \mu, F(dx))$, where σ^2 is the (instantaneous) variance of the BM component, $F(dx)$ is the Lévy density (density of jumps), and $\mu \in \mathbb{R}$. For $(\alpha, \beta) \subset \mathbb{R} \setminus 0$, $F((\alpha, \beta))dt$ is the probability of a jump from 0 into (α, β) during an infinitesimally small time interval dt.

Let X be the Lévy process on $\mathbb{R}$ with the generating triplet $(\sigma^2, \mu, F(dx))$ and the Lévy exponent Ψ under a probability measure $\mathbb{Q}$ on the filtered space generated by X. If $\int_{\mathbb{R}\setminus 0} F(dx) < \infty$, which means that the jump component of the process is a compound Poisson process, or, more generally, if $\int_{\mathbb{R}\setminus 0} \min\{|x|, 1\}F(dx) < \infty$, that is, the jump component is of finite variation, then there exist $\sigma^2 \geq 0$ and $\mu \in \mathbb{R}$ (variance and drift of the Brownian motion component of X) such that, for all β on the imaginary line $i\mathbb{R}$, where $i = \sqrt{-1}$ is the imaginary unit,

$$\Psi(\beta) = \frac{\sigma^2}{2}\beta^2 + b\beta + \int_{\mathbb{R}\setminus 0}(e^{\beta y} - 1)F(dy). \tag{A.1}$$

This is a special case of the Lévy-Khintchine formula (see., e.g., [46, Thm. 8.1]).

If $\mathbb{E}^{\mathbb{Q}}\left[e^{X_1}\right] < \infty$, equivalently, $\Psi(1)$ is well-defined and finite, then (A.1) is valid for all β in the strip $\{\beta \mid \operatorname{Re}\beta \in [0, 1]\}$. If there are no jumps, we obtain the Lévy exponent of the BM with drift $\Psi(\beta) = \sigma^2\beta^2/2 + \mu\beta$.

Let T_q be an exponentially distributed random variable $T_q \sim \operatorname{Exp} q$ of mean $1/q$, independent of X. Introduce the notation

$$\kappa_q^+(\beta) = \mathbb{E}^{\mathbb{Q}}\left[e^{\beta \overline{X}_{T_q}}\right], \quad \kappa_q^-(\beta) = \mathbb{E}^{\mathbb{Q}}\left[e^{\beta \underline{X}_{T_q}}\right]. \tag{A.2}$$

The Wiener-Hopf factorization formula states that, for $\beta \in i\mathbb{R}$,

$$q/(q - \Psi(\beta)) = \kappa_q^+(\beta)\kappa_q^-(\beta). \tag{A.3}$$

Note that the equality holds for β in the region each function in (A.3) admits the analytic continuation to. If $q - \Psi(1) > 0$, then this region contains a strip $\{\beta \mid \operatorname{Re}\beta \in [0, 1]\}$.

Example A.1. Consider the BM model with embedded positive jumps, where the density of jumps is given by the following Erlang distribution:

$$F_+(c_+, \lambda_+, dx) = \mathbb{1}_{(0,+\infty)}(x)c_+\lambda_+^2 xe^{-\lambda_+ x}dx, \tag{A.4}$$

where $c_+ > 0$ is the intensity of positive jumps, and $2/\lambda_+ > 0$ is the average size of a jump. There are no negative jumps. The Lévy exponent corresponding to the jump part is

$$\Psi_+(\beta) = \int_0^\infty \left(e^{\beta x} - 1\right) c_+\lambda_+^2 xe^{-\lambda_+ x}dx = c_+\left[\frac{\lambda_+^2}{(\lambda_+ - \beta)^2} - 1\right].$$

Adding $\Psi_+(\beta)$ to $\sigma^2\beta^2/2 + \mu\beta$, we obtain the Lévy exponent (**??**). The condition $\Psi(\beta) < \infty$ for any $\beta \in [0, 1]$ is equivalent to $\lambda_+ > 1$.

Let $q - \Psi(1) > 0$, and $\mu = -\sigma^2\lambda_+$. Then

$$\Psi(\beta) = \frac{\sigma^2}{2}\beta(\beta - 2\lambda_+) + \frac{c_+(\lambda_+^2 - (\lambda_+ - \beta)^2)}{(\lambda_+ - \beta)^2} = \beta(\beta - 2\lambda_+)\left(\frac{\sigma^2}{2} - \frac{c_+}{(\lambda_+ - \beta)^2}\right).$$

It is easy to see that the function Ψ has four zeroes: $\beta = 0$, $\beta = 2\lambda_+$, and $\beta = \lambda_+ \pm \sqrt{2c_+/\sigma^2}$. Assume that the first moment of the distribution of X_1 is positive. This is equivalent to $\Psi'(0) > 0$. Direct calculations show that $\Psi'(0) = \frac{\sigma^2}{\lambda_+}\left(\frac{2c_+}{\sigma^2} - \lambda_+^2\right)$. Hence, $\Psi'(0) > 0$ iff $\lambda_+ < \sqrt{2c_+/\sigma^2}$. Therefore, the function Ψ has only one negative zero, and two positive zeroes. Hence the characteristic equation $q - \Psi(\beta) = 0$ has one negative and three positive solutions,

$$\beta_q^- < \lambda_+ - \sqrt{\frac{2c_+}{\sigma^2}} < 0 < 1 < \beta_{q,1}^+ < \lambda_+ < \beta_{q,2}^+ < 2\lambda_+ < \lambda_+ + \sqrt{\frac{2c_+}{\sigma^2}} < \beta_{q,3}^+.$$

A.2. **Proof of Theorem 3.7.** (i) The sample paths of a subordinator are non-decreasing (and strictly increasing if $\mu > 0$). Furthermore, $\widetilde{V}(x)$ is non-increasing, hence, for any x, $T \mapsto \mathbb{E}^{\mathbb{Q},x}\left[\widetilde{V}(X_T)\right]$ is non-increasing. By definition, $\widetilde{V}(x) = 0$ for any $x \geq \bar{h}$, and $\widetilde{V}(x) \geq 0$ for any $x \in \mathbb{R}$. Therefore,

$$\mathcal{U}(\mu, \lambda; 0, x) = (1-\alpha)R - \lambda e^x + \mu\widetilde{V}(x) \geq (1-\alpha)R - \lambda e^x > 0, \text{ for } x < \bar{h}^0. \tag{A.5}$$

Hence $\bar{h} \geq \bar{h}^0$. Suppose that $\bar{h} > \bar{h}^0$. Then

$$\mathcal{U}(\mu, \lambda; 0, \bar{h}) = (1-\alpha)R - \lambda e^{\bar{h}} + \mu\widetilde{V}(\bar{h}) = (1-\alpha)R - \lambda e^{\bar{h}} < 0.$$

By continuity, there exists $\epsilon > 0$ such that $\bar{h} - \epsilon > \bar{h}^0$, and for $x \in (\bar{h} - \epsilon, \bar{h})$, $\mathcal{U}(\mu, \lambda; 0, x) < 0$. Since $\Psi(1) > 0$ and X is a subordinator, the function $\mathcal{U}(\mu, \lambda; T, x)$ given by (3.6) is strictly decreasing in T. Therefore, $\mathcal{U}(\mu, \lambda; T, x) < 0$ for all $T \geq 0$, hence $T_*(x) = 0$, which contradicts the definition of $\bar{h}$. Therefore $\bar{h} = \bar{h}^0$.

(ii) Since $X_T \to \infty$ as $T \to \infty$ a.s., and $\widetilde{V}(x) = 0$ for $x \geq \bar{h}$, the last term on the RHS of (3.6) vanishes as $T \to +\infty$. It is easy to see that for any $x \in \mathbb{R}$, $\mathcal{U}(\lambda; T, x) \to -\infty$ as $T \to \infty$. On the strength of (A.5), $\mathcal{U}(\lambda; 0, x) > 0$ for $x \leq \bar{h}^0$. Moreover, $\mathcal{U}(\lambda; T, x)$ is strictly decreasing in T because $\Psi(1) > 0$ and X is a subordinator. Therefore, for $x \leq \bar{h} = \bar{h}^0$, the equation $\mathcal{U}(\lambda, T, x) = 0$ has a unique solution $T_*(x) \in (0, +\infty)$.

(iii) Suppose $T_*(\bar{h} - 0) > 0$, then, by (ii), $\mathcal{U}(\lambda, T_*(\bar{h} - 0), \bar{h} - 0) = 0$. At the same time,

$$\mathcal{U}(\lambda; T_*(\bar{h} - 0), \bar{h} - 0) = (1-\alpha)R - \lambda e^{T_*(\bar{h}-0)+\bar{h}} + \lambda \lim_{x \uparrow \bar{h}} \mathbb{E}^{\mathbb{Q},x}\left[\widetilde{V}(X_{T_*(\bar{h}-0)})\right].$$

By the stochastic continuity of X, the limit on the RHS is zero. Hence

$$\mathcal{U}(\lambda; T_*(\bar{h} - 0), \bar{h} - 0) = (1-\alpha)R - \lambda e^{T_*(\bar{h}-0)+\bar{h}} < 0,$$

which is a contradiction. Hence, $T_*(\bar{h} - 0) = 0$.

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